

Online Appendix to Pre-empting Competitive Sales

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This appendix studies questions beyond the main paper’s existence and welfare results: further equilibrium regimes, disclosure, risk aversion, and alternative bargaining protocols. Notation follows the main paper. The buyer’s value is b , its resolution benefit is d , and $s \in \{L, H\}$ indexes the seller’s continuation state. The main paper contains the proofs needed for its own results. References to its numbered statements and equations are marked “main paper” where needed; hyperlinks refer to the companion PDF.

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OA1 Further Equilibrium Regimes and Information Structures

The main paper shows that probing and knockout can coexist. We first ask whether costly attempts can occur without rejection, and then examine selection, alternative seller information, and the local family of equilibria.

OA1.1 Equilibria without Rejection

The remaining support, $\{W, K\}$, combines waiting with an offer accepted in both seller states. Section OA1.4 gives its boundary-price construction: the knockout price must reproduce seller H 's continuation value in the selected pool, and both distant and infinitesimal probing deviations must be deterred. These checks are additional to the existence of a price fixed point. The following result excludes this support after D1 refinement under the aligned continuation state, including nonboundary-price constructions.

Proposition OA1.1 (No refined waiting–knockout equilibrium under the aligned state). *Suppose early and late match values have a common continuous CDF with positive density on $[b, \bar{b}]$. Maintain the aligned composite terminal specification: Q_H nontrivially first-order stochastically dominates Q_L and $v_L < v_H$. Suppose $\pi \in (0, 1)$, $\kappa > 0$, urgency is independent of b and has full interval support $[0, \bar{d}]$, the seller state is independent of buyer type, and Assumption 1 holds. Every pure PBE with action support $\{W, K\}$ fails criterion D1, in its strict-gain form, in the corresponding contingent-plan representation.*

The support $\{W, K\}$ can exist in a PBE before refinement, so D1 drives the exclusion. The same conclusion holds under late-participation variation alone. Under fallback variation, however, a top willingness plateau can support a refined $\{W, K\}$ equilibrium: above v_H , the higher no-sale continuation no longer changes buyer willingness. Section OA1.5 proves the proposition and gives this counterexample. Under urgency variation the willingness gap is flat everywhere, and an exact survival condition determines when $\{W, K\}$ survives. For a knockout price q_H , let $b_K^0 = \min\{\bar{b}, w^{-1}(q_H + \kappa)\}$ denote the highest feasible wait–knockout marginal value.

Proposition OA1.2 (D1 survival conditions for waiting–knockout under urgency variation). *Maintain the urgency-only specification with $\kappa > 0$ and $d_{S,L} > d_{S,H} \geq 0$, and fix a pure PBE with support $\{W, K\}$ and price q_H . The equilibrium has a D1-compatible completion in the contingent-plan representation if and only if*

$$d_{S,L} - d_{S,H} \leq C_H(b_K^0) - q_H + \frac{\pi\kappa}{1 - \pi} \quad \text{and} \quad q_H \leq C_H(b_K^0).$$

When $b_K^0 < \bar{b}$, the first condition reads $d_{S,L} \leq \Gamma(b_K^0) + \kappa/(1 - \pi)$, and it implies the second whenever $\Delta_D \geq 0$.

Section OA1.6 proves both directions and reports an interval-certified numerical instance. At the primitives of Example OA1.4, the boundary price $q_H = .637358$ satisfies both survival conditions, with margins .003044 and .013044. D1 therefore permits a refined $\{W, K\}$ equilibrium alongside the $\{W, P, K\}$ equilibrium at these urgency primitives. At these primitives, the survival boundary is $d_{S,L}^* = .023044$; on $(.023044, .060642]$, raw $\{W, K\}$ PBEs exist but fail D1. Conditional on positive attempt mass, refinement therefore makes rejections necessary under aligned or late-participation information, whereas fallback and urgency information can preserve attempts without rejection.

OA1.2 Selection and the Information in Rejection

In supports containing probing, the first attempt cutoff is strictly decreasing in b . At differentiability points,

$$\frac{\partial D_L(b; q_L)}{\partial b} = -w'_L(b) < 0. \quad (\text{OA1.1})$$

Consequently, in $\{W, P\}$ and $\{W, P, K\}$, attempt probability is weakly increasing in b and strictly increasing wherever the cutoff is interior; at differentiability points with an interior cutoff,

$$\frac{\partial \Pr(p > 0 \mid b)}{\partial b} = g(D_L(b))w'_L(b) > 0. \quad (\text{OA1.2})$$

The same monotonicity holds in $\{W, K\}$ because $\partial \widetilde{D}_H(b; q_H)/\partial b = -\bar{w}'(b) < 0$. Under independence of match values and urgency, observing an attempt therefore shifts the seller's posterior toward higher b : a buyer with a better match can justify the fixed offer cost and equilibrium price at a lower level of urgency.

Within $\{W, P, K\}$, the switch from probing to knockout is also reached at lower urgency by higher-value buyers. At differentiability points,

$$\frac{\partial D_H(b; q)}{\partial b} = -w'_H(b) < 0. \quad (\text{OA1.3})$$

The effect of b on probing itself is generally ambiguous, because a higher value both enters the probing region at its lower boundary and exits it toward knockout at its upper boundary. At differentiability points with interior cutoffs,

$$\frac{\partial \Pr(p = q_L \mid b)}{\partial b} = g(D_L(b))w'_L(b) - g(D_H(b))w'_H(b). \quad (\text{OA1.4})$$

Thus probing becomes more likely with b exactly when the inflow at $D_L(b)$ exceeds the outflow at $D_H(b)$. If urgency is uniform on $[0, \bar{d}]$ and both cutoffs are interior, then

$$\frac{\partial \Pr(p > 0 \mid b)}{\partial b} = \frac{w'_L(b)}{\bar{d}} > 0, \quad \frac{\partial \Pr(p = q_L \mid b)}{\partial b} = \frac{w'_L(b) - w'_H(b)}{\bar{d}} \leq 0, \quad (\text{OA1.5})$$

where the inequality follows from $w'_H(b) \geq w'_L(b)$. For any maintained positive urgency density, attempt incidence is weakly increasing in b and strictly increasing wherever the cutoff is interior; the shape of G determines the strength of this selection.

The buyer's terminal surplus is $u_s(b) = \mathbb{E}_s[\max\{b - \max\{v_s, Y_{1N}\}, 0\}]$. Under late-participation variation the state enters through Y_{1N} , so rejection of a probe is bad news about the round the rejected buyer now enters. Under fallback variation the state matters only when the fallback prevents the buyer from clearing the terminal threshold or sets the price under weak late competition; above v_H , the willingness gap is constant in b . Under urgency variation the state never enters u_s : rejection reveals a patient seller but leaves the buyer's auction payoff unchanged.¹

Selection also changes the interval of prices that separates the seller states. Conditional on probing, this interval has width $\int [C_H(b) - C_L(b)] h_P(b) db$. Under urgency variation, the gap is the constant $d_{S,L} - d_{S,H}$, so selection moves both thresholds equally. Under fallback variation, the gap declines with b , so high-value selection narrows it, though it remains bounded below by $\int_{v_L}^{v_H} \phi(F(t)) dt$. Under late-participation variation, high-value selection can instead preserve or widen the gap: above the common fallback, the derivative of $R_H(b) - R_L(b)$ is $[1 - F(b)][\phi'_H(F(b)) - \phi'_L(F(b))]$, which is positive for sufficiently large $F(b)$.

The ban comparison in the main paper, Section 4 applies to all three microfoundations. Disclosure likewise applies when the continuation state is verifiable; its numerical application isolates participation information, for which market-level certification is especially natural.

The additional aligned calibration combines stronger late participation, a higher fallback, and lower seller urgency in the high-continuation state, retaining their distinct economic roles.

OA1.3 Seller Urgency Alone

With seller urgency alone, $Q_L = Q_H$ and $v_L = v_H$, so $u_L = u_H$ and $w_L = w_H$. Screening operates through $C_s = R - d_{S,s}$, giving the following explicit probing window.

¹With continued bargaining, seller patience can also affect negotiation before the round.

Corollary OA1.3 (Urgency-only probing window). *Let Q and v be common across states, let $d_{S,L} > d_{S,H} \geq 0$, and write $\Delta z = d_{S,L} - d_{S,H}$ and Γ for the common wedge. Then Assumption 1 holds, with*

$$\bar{D}_L = \frac{\kappa}{1-\pi} - d_{S,L}, \quad \Delta_D = \Delta z - \frac{\pi\kappa}{1-\pi}.$$

Suppose

$$\frac{\pi\kappa}{1-\pi} < \Delta z, \quad d_{S,L} < \min\left\{R(\underline{b}), \Gamma(\underline{b}) + \frac{\kappa}{1-\pi}\right\}, \quad \bar{D}_L < \bar{d} \leq \frac{(1-\pi)d_{S,L} - d_{S,H}}{\pi}.$$

Then Proposition 3.1 applies with support $\{W, P\}$: no early offer, a rejected attempt, and successful pre-emption occur with positive probability, the probing offer being accepted exactly by the urgent seller. If in addition $d_{S,L} \leq \kappa/(1-\pi)$, the equilibrium has a D1-compatible completion.

Proof. With common (Q, v) , $w_H = w_L = w$ and $C_s = R - d_{S,s}$, so $C_H - C_L = \Delta z$ pointwise and Assumption 1 holds once $d_{S,L} < R(\underline{b})$. Since $\Gamma(\bar{b}) = 0$, $C_s(\bar{b}) = w(\bar{b}) - d_{S,s}$; substituting into (3.5) and (3.5) gives the displayed $\bar{D}_L$ and Δ_D , and $\bar{D}_L + \Delta_D/\pi = [(1-\pi)d_{S,L} - d_{S,H}]/\pi$. The second primitive bound is $\chi_L > 0$; together with $\Delta z > \pi\kappa/(1-\pi)$, it also implies $d_{S,H} < \Gamma(\underline{b}) + \kappa$, or $\chi_H > 0$. The D1 statement is the sufficient condition (C.1) of Corollary OA1.5. $\square$

The urgency gap must exceed the screening premium $\pi\kappa/(1-\pi)$ for a state-splitting price to exist, while the urgent seller's cost must stay below the level at which even the least attractive buyer could buy it out. The window's upper endpoint is positive exactly when $(1-\pi)d_{S,L} > d_{S,H}$, which the first condition already implies whenever $d_{S,H} \leq \kappa$. Beyond the displayed sufficient condition, Corollary OA1.5 gives the exact D1 requirement $\Gamma(b_L^0) \geq d_{S,L} - \kappa/(1-\pi)$ at an interior marginal value b_L^0 . The urgency-only specification can therefore support both $\{W, P\}$ and $\{W, P, K\}$. The one-price support admits the explicit primitive window above; once knockout is active, the two prices and selected buyer pools must be determined jointly. The following calibration verifies the richer full-menu support.

Example OA1.4 (Private seller urgency as the screening state). Let $F = U[0, 1]$, let the common terminal environment have $n = 2$ late bidders and no-sale continuation value $v = .4$, and let seller urgency be $d_{S,s} \in \{0, .02\}$ with equal probabilities, state L being the urgent seller. Let buyer urgency have the rate-10 truncated-exponential distribution on $[0, 1]$, and set $\kappa = .01$. An endpoint-inclusive numerical assessment has

$$(q_L, q_H) = (.586757, .643708)$$

and probabilities

$$(.550185, .092644, .357171)$$

of no early offer, a rejected attempt, and successful pre-emption. It satisfies all audited PBE and criterion-D1 conditions in the seller-plan representation.

The lower price is accepted only by the urgent seller, while the higher price is accepted by both types. Seller urgency can therefore reproduce the baseline screening pattern even without private information about late-participation prospects. The equilibrium equations are analytical; the example's fixed point and cross-type rankings are numerical. Section OA1.7 gives the proof and audit.

OA1.4 Waiting–knockout construction

The remaining one-price support is $\{W, K\}$: some buyers wait, while the others make a knockout offer accepted in both seller states. It produces attempts and completed pre-emption but no rejected attempts, so the seller state does not generate different on-path responses, although it still affects the equilibrium price and buyer cutoff.

The waiting–knockout cutoff and conditional action probability are given in (B.5).

Buyers with $d < \widetilde{D}_H(b; q)$ wait, while those with $d \geq \widetilde{D}_H(b; q)$ make the knockout offer; hence $m_K(b; q)$ is the probability of knockout conditional on b . Whenever the aggregate knockout probability is positive, seller H 's expected continuation value after observing the offer is

$$T_K(q) = \frac{\int_{\underline{b}}^{\bar{b}} C_H(b) m_K(b; q) f(b) db}{\int_{\underline{b}}^{\bar{b}} m_K(b; q) f(b) db}. \quad (\text{OA1.6})$$

The equality $q = T_K(q)$ is the waiting–knockout boundary-price condition: seller H is indifferent between accepting the knockout price and continuing to the bidding round. It is the price selected by Assumption 2 when knockout is the active accepted-offer category. Two further checks ensure that both actions occur and that no buyer prefers an unused probing offer. First,

$$0 < \widetilde{D}_H(\underline{b}; q) \quad \text{and} \quad \widetilde{D}_H(\bar{b}; q) < \bar{d}. \quad (\text{OA1.7})$$

Because the cutoff decreases with b , these inequalities leave fully patient, low-value buyers who wait and sufficiently urgent, high-value buyers who choose knockout. Second, for every buyer type (b, d) , require

$$(1 - \pi)[w_L(b) + d - C_L(\bar{b})] - \kappa \leq \max\{0, \bar{w}(b) + d - q - \kappa\}. \quad (\text{OA1.8})$$

Under the off-path belief $b = \bar{b}$, $C_L(\bar{b})$ is the cheapest price seller L would accept. The left side of (OA1.8) is the buyer's payoff from that cheapest probe; the right side is its best

payoff from waiting or the on-path knockout. The inequality therefore keeps probing out of the buyer's optimal action set.

Local passivity adds a separate nearby-deviation check. An infinitesimal undercut inherits the knockout pool's posterior, so seller H rejects it while seller L accepts it; the undercut is therefore a probe rather than a cheaper knockout. A strict sufficient condition for deterring these local probes is

$$(1 - \pi)[w_L(b) + d - q] - \kappa < \max\{0, \bar{w}(b) + d - q - \kappa\} \quad \text{for every } (b, d). \quad (\text{OA1.9})$$

At a fixed point satisfying (OA1.7)–(OA1.9), Bayes' rule makes seller H indifferent at q , and state ordering makes seller L accept. After sufficiently small undercuts, assign the on-path knockout posterior; the strict inequality (OA1.9) deters the resulting local probes. At all remaining unused prices, assign $b = \bar{b}$. Any unused offer accepted in both states must then cost at least $C_H(\bar{b}) \geq q$, while (OA1.8) deters an offer accepted only in L . Buyers wait below $\widetilde{D}_H(b; q)$ and offer q above it. These strategies and beliefs form a locally passive PBE with support $\{W, K\}$.

The fixed point exists under simple endpoint conditions. For every $q \in [C_H(b), C_H(\bar{b})]$, $T_K(q)$ is a weighted average of $C_H(b)$ and therefore lies in the same interval. Moreover, $\chi_H > 0$ gives $\widetilde{D}_H(b; C_H(b)) > 0$, while $C_H(\bar{b}) - \bar{w}(\bar{b}) + \kappa < \bar{d}$ gives $\widetilde{D}_H(\bar{b}; C_H(\bar{b})) < \bar{d}$. These endpoint inequalities imply positive waiting and knockout probabilities throughout the interval, so T_K is continuous and has a fixed point. A root implements the selected $\{W, K\}$ support when it also satisfies (OA1.8) and (OA1.9). Unlike the $\{W, P\}$ result, these additional checks do not reduce to a single interval for $\bar{d}$: they must deter both distant and infinitesimal probing offers for every buyer type. For example, holding $\bar{d}$ and the waiting–knockout fixed point fixed, a lower continuation value for seller L can make an unused probing offer attractive to intermediate buyer types and invalidate the $\{W, K\}$ support.

Pre-emption can therefore be driven by gains from certain early resolution without any on-path offer being used to screen the seller state. Whether this rejection-free outcome survives criterion D1 depends on what the state changes. When the state changes late-participation prospects, it also changes the buyer's continuation payoff, and D1 rules out the $\{W, K\}$ equilibrium. When the state changes only the seller's fallback or urgency, buyer willingness may be locally insensitive to that information, allowing the equilibrium to survive under the conditions in Section OA1.1.

OA1.5 Proof of Proposition OA1.1

Let q_H be the knockout price and define the wait–knockout cutoff

$$D_K(b) = q_H - \bar{w}(b) + \kappa, \quad \mathcal{B}_K = \{b : 0 \leq D_K(b) \leq \bar{d}\}, \quad \beta_K = \inf \mathcal{B}_K.$$

Positive waiting and knockout mass, continuity, and full interval urgency support make this marginal set nonempty, and positive knockout mass gives $\beta_K < \bar{b}$. Write $A = 1 - \pi$, $\Delta(b) = w_H(b) - w_L(b)$, and consider an undercut $p = q_H - \delta$, where $\delta > 0$. By the common-posterior ordering, every rationalizable seller plan $a = (a_L, a_H)$ is nested and belongs to

$$\mathcal{A}(p) \subseteq \{(x, 0) : x \in [0, 1]\} \cup \{(1, y) : y \in [0, 1]\}.$$

Let $\rho(a) = Aa_L + \pi a_H$. At the hypothetical wait-knockout marginal type $(b, D_K(b))$, the deviation gain is

$$m_b(a) = \rho(a)(\delta + \kappa) - A\pi(a_L - a_H)\Delta(b) - \kappa. \quad (\text{OA1.10})$$

For a type with urgency $d = D_K(b) + z$, the equilibrium payoff is $\max\{z, 0\}$, so its deviation gain is exactly

$$h_{b,z}(a) = m_b(a) + \rho(a)z - \max\{z, 0\}. \quad (\text{OA1.11})$$

Under the aligned composite specification, equation (A.3) gives

$$\Delta(b) = 0 \quad \text{for } b \leq v_L, \quad \Delta'(b) = \begin{cases} \phi_L(F(b)) > 0, & v_L < b < v_H, \\ \phi_L(F(b)) - \phi_H(F(b)) > 0, & b > v_H. \end{cases}$$

In the late-participation-only specialization $v_L = v_H = v_0$, the same expressions hold with the middle branch empty, and the argument below applies verbatim. Define

$$\mathcal{B}_K^* = \begin{cases} \{\beta_K\}, & \beta_K \geq v_L, \\ \mathcal{B}_K \cap [\beta_K, v_L], & \beta_K < v_L. \end{cases}$$

For any $b_* \in \mathcal{B}_K^*$, every $b \geq \beta_K$, and every nested plan,

$$m_{b_*}(a) - m_b(a) = A\pi(a_L - a_H)[\Delta(b) - \Delta(b_*)] \geq 0. \quad (\text{OA1.12})$$

The inequality is strict at a partial plan whenever $b > v_L$ and $b_* \leq v_L$, or whenever $b > b_* \geq v_L$. Marginal types in $\mathcal{B}_K^*$ have identical gain sets when $\beta_K < v_L$. At full acceptance all marginal types gain the same $\delta > 0$. Moreover, $\rho z - \max\{z, 0\} \leq 0$, with strict inequality whenever a nonmarginal type could otherwise tie at a payoff-relevant partial plan. Thus the marginal type at a given b is weakly most willing.

If $b < \beta_K$, then $D_K(b) > \bar{d}$. When $\beta_K > \underline{b}$, continuity gives $D_K(\beta_K) = \bar{d}$, and the deviation payoff is strictly increasing in both b and d whenever $\rho > 0$; rejection alone yields $-\kappa < 0$. Hence the same strict comparison holds below β_K . If $\beta_K = \underline{b}$, that region is empty. Equations (OA1.11)–(OA1.12) therefore leave the single marginal type

$(\beta_K, D_K(\beta_K))$ D1-undominated when $\beta_K \geq v_L$. When $\beta_K < v_L$, the D1-undominated set is

$$\Theta_K^* = \{(b, D_K(b)) : b \in \mathcal{B}_K^*\};$$

these plateau types tie rather than delete one another. On the branch $(1, y)$, their common marginal gain is zero at

$$y_K^* = 1 - \frac{\delta}{\pi\{\delta + \kappa + A\Delta(\beta_K)\}} \in (0, 1)$$

for δ sufficiently small, while no type outside the surviving set gains strictly there. Hence no type in the set is itself deleted.

Positive knockout mass and strict increase of $\bar{w}$ give the on-path pool positive probability above $\max\{\beta_K, v_H\}$. Since C_H is constant below v_H and strictly increasing above it, seller H 's on-path optimality gives

$$C_H(\beta_K) < \mathbb{E}[C_H(b) \mid q_H] \leq q_H.$$

Moreover, $q_H \leq C_H(\bar{b})$: otherwise a slightly lower price above $C_H(\bar{b})$ would be accepted in both states under every posterior and would strictly dominate q_H for its users. Because $C_H(\beta_K) > C_L(\bar{b}) > 0$, choose a positive $p \in (C_H(\beta_K), q_H)$ sufficiently close to q_H . Continuity and strict increase above v_H give a unique $b_p > v_H$ satisfying $C_H(b_p) = p$. A posterior concentrated on b_p makes seller H indifferent and seller L strictly willing to accept, so the whole branch $(1, y)$ is rationalizable. Strict-gain D1 restricts the buyer posterior to the singleton marginal type if $\beta_K \geq v_L$, and to Θ_K^* if $\beta_K < v_L$. In the latter case C_H has the same value $C_H(\beta_K)$ at every permitted type. Every permitted posterior therefore makes both sellers accept p , and every surviving marginal buyer saves $q_H - p > 0$. The $\{W, K\}$ assessment fails D1 in the contingent-plan representation.

Fallback-only boundary. The exclusion is false without late-participation variation. Let $F = U[0, 1]$, let both states have exactly two late buyers, $(v_L, v_H) = (0.2, 0.4)$, $\pi = 1/2$, $d_S = 0.01$, $\kappa = 0.05$, and $d \sim U[0, 0.15]$. The boundary knockout map has a numerically bracketed fixed point $q_H = 0.654466963$, with lowest marginal value $\beta_K = 0.623111832 > v_H$ and knockout mass 0.204305347. For every $b \geq v_H$, $w_H(b) - w_L(b) = 7/375$, so all feasible marginal types have the same D1 gain set. Belief on the top marginal type makes both sellers reject every undercut because $C_L(1) = 989/1500 > q_H$; no overcut is profitable. Hence this fallback-only knockout equilibrium has positive waiting and knockout mass and admits a D1-compatible completion in the contingent-plan representation.

Parametric diagnostic. Specialize to the late-participation-only case $v_L = v_H = 0$. In the uniform benchmark $b \sim U[0, 1]$, $(n_L, n_H) = (2, 7)$, $\pi = 1/2$, and $d_S = 0.01$, the restrictions

$$\bar{d} > \frac{319}{600}, \quad \frac{7}{16} < \kappa < \bar{d} - \frac{113}{1200}$$

support a nondegenerate boundary-price PBE with support $\{W, K\}$ before refinement. Indeed, the knockout posterior map continuously maps $[C_H(0), C_H(1)]$ into itself and the upper inequality gives positive knockout mass. The lower inequality also gives

$$\max_b Z(b; C_L(1), q_H) \leq C_H(1) - C_L(1) - \kappa = \frac{5}{24} - \kappa < 0,$$

so the top-value off-path belief deters probing. The proposition then eliminates this raw equilibrium. This appendix calculation is only a diagnostic; the general result does not cover mixed acceptance or an all-knockout assessment.

OA1.6 Proof of Proposition OA1.2

Maintain the urgency-only specification: common (Q, v) , hence a common w , R , and wedge $\Gamma = R - w$, with $C_s = R - d_{S,s}$ and $d_{S,L} > d_{S,H} \geq 0$. Fix a pure PBE with support $\{W, K\}$ and price q_H , so a buyer attempts exactly when $w(b) + d \geq q_H + \kappa$ and both seller states accept q_H . Note first that the first survival condition is equivalent to

$$C_L(b_K^0) \geq q_H - \frac{\pi\kappa}{1-\pi}, \quad (\text{OA1.13})$$

since $C_L = C_H - (d_{S,L} - d_{S,H})$ pointwise; when $b_K^0 < \bar{b}$, the marginal identity $w(b_K^0) = q_H + \kappa$ gives $C_s(b_K^0) = q_H + \kappa + \Gamma(b_K^0) - d_{S,s}$, so (OA1.13) reads $d_{S,L} \leq \Gamma(b_K^0) + \kappa/(1-\pi)$, and $\Delta_D \geq 0$ then delivers the second condition because $d_{S,H} \leq d_{S,L} - \pi\kappa/(1-\pi) \leq \Gamma(b_K^0) + \kappa$.

Scalar reduction and the tied curve. Because $w_H \equiv w_L$, a contingent plan $a = (a_L, a_H)$ matters to a deviating buyer only through its acceptance weight $\rho = (1-\pi)a_L + \pi a_H \in [0, 1]$: a type with scalar $z = w(b) + d$ obtains $\rho(z - p) - \kappa$ from deviating to p , against the equilibrium value $U^*(z) = \max\{0, z - q_H - \kappa\}$. For $p < q_H$ the weak-gain set is $\{\rho \geq \tau(z; p)\}$ with $\tau(z; p) = [\kappa + U^*(z)]/(z - p)$, strictly decreasing in z below $q_H + \kappa$ and strictly increasing above it. The deviation-eager types after every undercut therefore form the tied curve $\Theta_K = \{(b, d) : w(b) + d = q_H + \kappa\}$, whose value range is $[\beta_K, b_K^0]$ with $b_K^0 = \min\{\bar{b}, w^{-1}(q_H + \kappa)\}$. All curve types have identical gain sets, and whenever some rationalizable plan gives the curve types a strict gain, the strict-gain comparison deletes every off-curve type. On the $(x, 0)$ branch this bite occurs exactly when $p < q_H - \pi\kappa/(1-\pi)$, so that $\tau(q_H + \kappa; p) < 1 - \pi$; on the $(1, y)$ branch it extends to every undercut $p \in [C_H(b), q_H]$, where a posterior making the patient state indifferent rationalizes an interior y . D1-permitted posteriors after an undercut are therefore the

distributions on Θ_K , and $\mathbb{E}[C_s(b)]$ over permitted posteriors is maximized at the singleton on the top curve type $(b_K^0, D_K(b_K^0))$.

Survival. Suppose both survival conditions hold. Place the off-path belief on $(b_K^0, D_K(b_K^0))$ at every $p \neq q_H$ and let each seller respond sequentially rationally, accepting at equality. For $p < C_L(b_K^0)$, both states reject and every type's gain is at most $-\kappa < 0$. For $p \in [C_L(b_K^0), q_H]$, the urgent state accepts and the patient state rejects, since $p < q_H \leq C_H(b_K^0)$ by the second condition; thus $\rho = 1 - \pi$, and the supremum over all types of $(1 - \pi)(z - p) - \kappa - U^*(z)$ is attained on the curve, where it equals

$$(1 - \pi)(q_H + \kappa - p) - \kappa \leq (1 - \pi)(q_H + \kappa - C_L(b_K^0)) - \kappa \leq 0$$

by (OA1.13). For $p > q_H$, every type's gain is negative under every plan, so gain sets are empty and D1 places no restriction. The belief is D1-permitted at every off-path price and no deviation strictly gains: the equilibrium survives. At equality in (OA1.13) the supremum is zero, weak rather than strict, so the strict-gain criterion still passes; the survival region is closed.

Failure. Suppose (OA1.13) fails. Pick $p \in (C_L(b_K^0), q_H - \pi\kappa/(1 - \pi))$, a nonempty interval. Deletion has bite and leaves exactly Θ_K , so under every permitted posterior $\mathbb{E}[C_L(b)] \leq C_L(b_K^0) < p$: the urgent state must accept, giving $\rho \geq 1 - \pi$, and every curve type gains at least $(1 - \pi)(q_H + \kappa - p) - \kappa > 0$, strictly. If instead the second condition fails, $q_H > C_H(b_K^0)$, any $p \in (C_H(b_K^0), q_H)$ forces acceptance by both states under every permitted posterior, and every knockout type gains $q_H - p > 0$. In either case every D1-consistent completion admits a profitable deviation, so the assessment fails strict-gain D1. $\square$

Numerical instance and boundary. At the primitives of Example OA1.4, the boundary fixed point for support $\{W, K\}$, $q_H = \mathbb{E}[C_H(b) \mid \text{attempt}]$, equals .637358061, with waiting mass .615537 and knockout mass .384463; $b_K^0 = .790989$ and $\Gamma(b_K^0) = .003044$, so both survival conditions hold with margins .003044 and .013044, and this refined equilibrium coexists with the Example's full-menu equilibrium. Because the fixed point does not involve $d_{S,L}$, the survival boundary there is the vertical cut $d_{S,L}^* = \Gamma(b_K^0) + \kappa/(1 - \pi) = .023044$: certified failures occur at $d_{S,L} \in \{.025, .03, .05\}$, while raw $\{W, K\}$ PBEs exist up to $d_{S,L} = .060642$, so on $(.023044, .060642]$ the exclusion is driven by criterion D1 exactly as in the aligned case. Raising the offer cost to $\kappa = .05$ yields a variant with $D_K(\bar{b}) \geq 0$ and $C_L(\bar{b}) > q_H = .643707$ (numerically close to, but distinct from, the Example's full-menu price .643708), in which the belief on the top-value marginal type makes both states reject every undercut—the exact analogue of the fallback-only construction above. Because $R' \leq w'$ above v , the first-condition residual $q_H - \pi\kappa/(1 - \pi) - C_L(b_K^0(q_H))$ is nondecreasing in q_H , so survival of any waiting–knockout price is decided at the lowest, boundary-priced fixed point, and

the D1-surviving price set is an interval. An outward-rounded ball-arithmetic calculation applies the Krawczyk theorem to both boundary fixed points—for $\kappa = .01$ in the marginal coordinate b_K^0 , and for $\kappa = .05$ directly in the price—and proves existence and local uniqueness with root enclosures of radius below 2×10^{-12} . The same calculation certifies strict signs, with enclosure radii below 10^{-11} , for both survival-condition margins, the survival boundary, the waiting and knockout masses, the partial-acceptance deterrence bound, the killer-interval widths and raw-persistence margins at $d_{S,L} \in \{.025, .03, .05\}$, and the corner and full-rejection margins of the $\kappa = .05$ variant. Every strategic margin used is certified strict except the intended boundary equalities.

OA1.7 Seller Urgency as the Screening State

Verification of Example OA1.4. Write $A = 1 - \pi = \pi = 1/2$. Because the terminal environment is common across seller types, write its buyer willingness and seller revenue as $w(b)$ and $R(b)$. The urgent and patient seller continuations are $R(b) - .02$ and $R(b)$. For candidate prices (q_L, q_H) , buyer optimality gives cutoffs

$$D_L(b) = q_L - w(b) + \frac{\kappa}{A}, \quad D_H(b) = \frac{q_H - Aq_L}{\pi} - w(b).$$

The buyer waits below D_L , probes between the cutoffs, and makes the knockout offer above D_H . Setting q_L equal to the urgent seller's posterior continuation in the probe pool and q_H equal to the patient seller's continuation in the knockout pool gives the displayed fixed point. A 131,072-panel endpoint-inclusive audit gives residuals below 1.7×10^{-14} , action masses

$$(.550185181, .185288486, .264526333),$$

and the positive constant cutoff gap .093902289. At q_L , the urgent seller is indifferent and the patient seller rejects with margin .02. At q_H , the patient seller is indifferent and the urgent seller accepts with the same margin.

For a raw off-path completion, belief on $b = 1$ gives continuation thresholds (.668, .688). Prices below .668 are rejected, prices in [.668, .688) buy only the urgent seller at a higher price than q_L , and prices weakly above .688 buy both types at a higher price than q_H . No deviation is profitable.

For D1, buyer payoffs depend on (b, d) only through $z = w(b) + d$. A contingent seller plan matters only through its aggregate acceptance probability r , and deviation payoff is $r(z - p) - \kappa$. For $.452 \leq p < q_L$, strict-gain sets are ordered by their crossing threshold

$$\frac{\kappa + U^*(z)}{z - p}, \quad U^*(z) = \max\{0, A(z - q_L) - \kappa, z - q_H - \kappa\},$$

which is minimized at $z_L = q_L + \kappa/A$. D1 leaves all types with this same z_L tied. Belief on their largest feasible value $b_L^0 = .699522891$ makes both sellers reject because

$$R(b_L^0) - .02 - q_L = .009043008 > 0.$$

For $q_L < p < q_H$, the crossing threshold is minimized at $z_H = (q_H - Aq_L)/\pi$. Its largest feasible value is $b_H^0 = 1$, and

$$R(1) - .02 - q_H = .024292340 > 0.$$

Both sellers again reject. Below .452, rejection is the only rationalizable seller plan, so every buyer's gain set is empty and the first belief remains D1-permitted. Above q_H , the on-path knockout is strictly cheaper than any deviation under the same or narrower acceptance. These regions exhaust the off-path price set and complete the D1 proof. $\square$

OA1.8 Probing refinement beyond the sufficient condition

For a boundary-price waiting-probing equilibrium, let $b_L^0 = \sup\{b : D_L(b; q_L) \geq 0\}$. The sufficient D1 condition in the main paper is part of the following sharper statement.

Corollary OA1.5 (D1 survival). *The on-path prices and strategies of the waiting-only PBE in Proposition 3.1 admit a D1-compatible belief completion in the contingent-plan representation. For $\kappa > 0$, the on-path profile of a boundary-price waiting-probing PBE in that proposition admits such a completion if and only if $q_L \leq C_L(b_L^0)$. The following parameter condition is sufficient:*

$$\bar{D}_L \geq 0. \tag{OA1.14}$$

The undercut argument in the main paper proves sufficiency. If $q_L > C_L(b_L^0)$, a price $p \in (C_L(b_L^0), q_L)$ is accepted by seller L under every surviving posterior and benefits the marginal types. The same strict-gain argument used for the disclosed-state result below establishes necessity. Overcuts are treated as in the main paper.

Boundary Pricing and Probing-Price Multiplicity

Boundary pricing selects a convenient zero-rent member of a larger PBE and D1-compatible set. An exact example uses the late-participation-only uniform specialization

$$(n_L, n_H) = (2, 7), \quad v_L = v_H = 0, \quad \pi = \frac{1}{2}, \quad d_S = .01, \quad \kappa = .05, \quad \bar{d} = .15,$$

and the rate-10 truncated-exponential urgency distribution. Besides the boundary fixed point, consider

$$\tilde{q}_L = C_L(1) = \frac{197}{300}.$$

The top-value cutoff is $D_L(1; \tilde{q}_L) = 9/100 < \bar{d}$, while sufficiently low values wait, so both actions have positive mass. The probing pool is nondegenerate below $b = 1$; strict monotonicity of C_L therefore gives

$$\mathbb{E}[C_L(b) \mid \tilde{q}_L] < C_L(1) = \tilde{q}_L,$$

and seller L accepts strictly. Seller H rejects because $\tilde{q}_L < C_H(0) = 37/50$.

Assign every off-path price to $b = 1$. An undercut is rejected because it is below $C_L(1)$, and an accepted L -only overcut is more expensive than the equilibrium probe. The cheapest both-state offer accepted under every seller posterior is $C_H(1)$. For the top type, its payoff relative to probing is

$$V_H(1, \bar{d}; C_H(1)) - V_L(1, \bar{d}; \tilde{q}_L) = -\frac{29}{1200} < 0,$$

and the difference is weakly smaller for every other buyer type. The complete contingent-plan comparison permits the same endpoint belief: here $b_L^0 = 1$ and the exact D1 condition holds at equality,

$$\tilde{q}_L = C_L(b_L^0).$$

All remaining seller-response, participation, and deviation inequalities are strict. Continuity therefore yields a nondegenerate interval of waiting-probing, D1-compatible prices immediately below $\tilde{q}_L$, with seller L earning a positive rent throughout. This establishes local price multiplicity under D1; these prices are outside the locally passive selection in Assumption 2.

OA1.9 Local distributional robustness

Let $\zeta = F^{-1}$ denote the common match-value quantile and T_ζ the associated price map. Call a quantile admissible if it has the maintained support and satisfies the continuity and positive-density assumptions. Holding the support endpoints, participation distributions, no-sale continuation values, and seller timing costs fixed, the full-menu fixed point in Proposition 3.2 persists under sufficiently small sup-norm perturbations of ζ within this class.

Corollary OA1.6 (Local distributional robustness and the surrounding PBE family). *Let q^0 be a fixed point under an admissible quantile ζ^0 at which the pointwise inequalities in (B.3) are strict. Define $\Psi(\zeta, q) = T_\zeta(q) - q$, and suppose the price Jacobian $D_q \Psi(\zeta^0, q^0)$ is nonsingular. Then, for every admissible quantile ζ sufficiently close to ζ^0 in the sup norm and every sufficiently small rent vector $r = (r_L, r_H) \in \mathbb{R}_+^2$, there is a unique nearby price pair $q(\zeta, r)$ satisfying*

$$q(\zeta, r) - T_\zeta(q(\zeta, r)) = r.$$

It implements a full-menu PBE satisfying (B.3) for every buyer value. In particular, $r = 0$ gives the locally unique nearby boundary-price equilibrium selected by Assumption 2.

Seller L 's marginal probing rent is $r_L = q_L - T_{\zeta,P}(q)$, and seller H 's marginal knockout rent is $r_H = q_H - T_{\zeta,K}(q)$. The selected boundary-price equilibrium sets both to zero. The positive-rent members show that the same action support and seller acceptance pattern persist locally in the larger PBE correspondence, but they are excluded by Assumption 2. Moving the rent vector also moves buyer cutoffs, selected buyer pools, and outcome probabilities, so the surrounding family is not a pure transfer of surplus. Section OA1.10 proves the local result. The aligned example in Section OA5.1 separately supplies an outward-rounded interval certificate for full-menu existence at the displayed primitives.

OA1.10 Proof of Corollary OA1.6

Write $b = \zeta(u)$, where u is uniform on $[0, 1]$, and hold (Q_L, Q_H, v_L, v_H) fixed. Let $(U_i)_{i \geq 1}$ be iid uniform draws, independent of $N \sim Q_s$, and let $U_{1N} \geq U_{2N}$ be the two highest among the first N draws, with $U_{1,0} = U_{2,0} = U_{2,1} = 0$. The auction objects can be written over this fixed probability space as

$$\begin{aligned} w_s^\zeta(u) &= \mathbb{E}[\min\{\zeta(u), \max\{v_s, \zeta(U_{1N})\}\}], \\ R_s^\zeta(u) &= \mathbb{E}[\max\{v_s, \zeta(U_{2N}), \min\{\zeta(u), \zeta(U_{1N})\}\}]. \end{aligned}$$

The min and max operators are Lipschitz. Hence w_s^ζ , R_s^ζ , and $C_s^\zeta = R_s^\zeta - d_{S,s}$ vary uniformly continuously with ζ under the sup norm, including when probability moves across either standing value.

In quantile coordinates, the action masses and both coordinates of T_ζ are integrals over a fixed uniform measure. Strict global interiority keeps all urgency cutoffs in a compact subset of $(0, \bar{d})$ in a neighborhood of (ζ^0, q^0) . Since G has a continuously differentiable positive density and both action masses are positive, $\Psi(\zeta, q) = T_\zeta(q) - q$ is continuous in (ζ, q) , continuously differentiable in q , and $D_q \Psi$ is jointly continuous. Differentiability with respect to the quantile is not required.

Apply the parameterized implicit-function theorem to

$$\Phi(\zeta, r, q) = T_\zeta(q) - q + r.$$

Nonsingularity of $D_q \Psi(\zeta^0, q^0)$ gives a unique nearby solution $q(\zeta, r)$ to $\Phi = 0$, continuous in (ζ, r) . At this solution,

$$q_L - T_{\zeta,P}(q) = r_L, \quad q_H - T_{\zeta,K}(q) = r_H,$$

so the relevant marginal seller accepts weakly, and strictly when the corresponding rent is positive. Every cutoff, cross-state seller response, positive-mass condition, and endpoint inequality used to deter off-path deviations is strict at $(\zeta^0, 0, q^0)$: positive action masses and strict monotonicity imply $q_L^0 = T_{\zeta^0, P}(q^0) < C_L^{\zeta^0}(\bar{b})$ and $q_H^0 = T_{\zeta^0, K}(q^0) < C_H^{\zeta^0}(\bar{b})$. These inequalities are continuous in (ζ, r, q) . The solution therefore remains globally interior and implements the buyer strategy and acceptance pattern in Proposition 3.2. For every admissible common distribution, pointwise $C_L < C_H$ nests seller responses. When $\kappa > 0$, the strict endpoint margins in the nested-plan argument in the main paper, Appendix C persist and supply the stated D1 completion in the contingent-plan representation.

At $r = 0$, the solution is the locally passive boundary-price branch. Positive rent vectors establish the surrounding PBE and D1 robustness result but violate Assumption 2 by the undercut argument following that assumption.

OA2 Disclosure of Verifiable Seller Information

Disclosure changes what the buyer knows before paying to offer. The results in this section require a specified verification technology: the seller can certify the relevant continuation state. They do not imply that subjective preferences, urgency, or deep value parameters can be truthfully certified in every application. Where only participation prospects can be verified, the analysis concerns that component and does not reveal the remaining private preferences.

OA2.1 Committed Disclosure of the Seller State

Suppose a verifiable signal identifies the seller's continuation state s before the buyer acts and therefore reveals the state-contingent objects $(Q_s, v_s, d_{S,s})$. The result below applies whether the ordering originates in late participation, fallback, seller urgency, or their aligned combination. The numerical application later in the section sets $v_L = v_H$ and $d_{S,L} = d_{S,H}$ to isolate disclosure of participation prospects. Maintain $C_H(b) > C_L(b)$.

Whereas a ban suppresses all early contracting, disclosure preserves the option of early agreement while revealing the seller's continuation state. Consider first an automatic disclosure regime: s becomes verifiably public before the buyer acts, either because reporting is mandatory or because the seller chose the disclosure technology before observing s . There is no interim reporting choice in this regime. Voluntary certification after observing s is analyzed separately below.

Disclosure removes one reason for rejection: the buyer knows which seller state it faces before deciding whether to incur the offer cost. With a positive attempt cost, an offer known to be rejected is strictly worse than waiting, so every on-path offer is accepted.

Early-sale incidence and welfare remain ambiguous because disclosure changes both prices and offer selection and removes the information rents an opaque seller earns from screening.

Conditional on state s , define

$$\widehat{D}_s(b; q) = q - w_s(b) + \kappa, \quad (\text{OA2.1})$$

$$\widehat{m}_s(b; q) = 1 - G(\widehat{D}_s(b; q)), \quad (\text{OA2.2})$$

and, whenever the denominator is positive,

$$\widehat{T}_s(q) = \frac{\int_{\underline{b}}^{\bar{b}} C_s(b) \widehat{m}_s(b; q) f(b) db}{\int_{\underline{b}}^{\bar{b}} \widehat{m}_s(b; q) f(b) db}. \quad (\text{OA2.3})$$

The map is the seller's posterior continuation value among buyers who offer at price q . Let

$$\bar{D}_s^{\text{FD}} = C_s(\bar{b}) - w_s(\bar{b}) + \kappa. \quad (\text{OA2.4})$$

Proposition OA2.1 (Automatic disclosure of the seller state). *Fix a disclosed state s under Assumption 1.*

(i) *A no-early-offer PBE exists if and only if*

$$\bar{d} \leq \bar{D}_s^{\text{FD}}, \quad (\text{OA2.5})$$

and it has a D1-compatible completion.

(ii) *If $\bar{d} > \bar{D}_s^{\text{FD}}$, then a no-early-offer PBE does not exist and $\widehat{T}_s$ has a fixed point $\hat{q}_s \in (C_s(\underline{b}), C_s(\bar{b}))$. There is a PBE in which the buyer waits below $\widehat{D}_s(b; \hat{q}_s)$, offers $\hat{q}_s$ above it, and the seller accepts.*

(iii) *If $\kappa > 0$, every on-path submitted offer is accepted in any pure-strategy PBE under disclosure of the seller state.*

(iv) *Suppose $\kappa > 0$ and the set of marginal values is nonempty. Define*

$$b_s^0 = \sup \{b : 0 \leq \widehat{D}_s(b; \hat{q}_s) \leq \bar{d}\}. \quad (\text{OA2.6})$$

The pooling PBE in part (ii) has a D1-compatible completion if and only if

$$\hat{q}_s \leq C_s(b_s^0). \quad (\text{OA2.7})$$

A sufficient condition is $\widehat{D}_s(\bar{b}; \hat{q}_s) \geq 0$.

Public disclosure reduces the problem to a state-specific pooling fixed point. With $\kappa > 0$, a knowingly rejected offer is strictly dominated by waiting; Section OA2.2 establishes existence and the D1 boundary.

Combine the two disclosed-state problems by using no-early-offer behavior in a state satisfying (OA2.5) and the pooling equilibrium otherwise. Write FD for this canonical assessment with full disclosure of the seller state, set $\widehat{M}_s = 0$ in a no-early-offer state, and otherwise let

$$\widehat{M}_s = \int_{\underline{b}}^{\bar{b}} \widehat{m}_s(b; \widehat{q}_s) f(b) db. \quad (\text{OA2.8})$$

Then

$$\Pr_{\text{FD}}(O) = \Pr_{\text{FD}}(A) = (1 - \pi)\widehat{M}_L + \pi\widehat{M}_H, \quad \Pr_{\text{FD}}(O \setminus A) = 0. \quad (\text{OA2.9})$$

Under opacity, early trade occurs above $D_L(b)$ in state L and above $D_H(b)$ in state H . Disclosure replaces these screening thresholds with the disclosed-state cutoff $\widehat{D}_s(b; \widehat{q}_s) = \widehat{q}_s - w_s(b) + \kappa$. A lower disclosed cutoff brings less urgent buyers—and, at a given urgency, buyers with lower match values—into early trade; a higher cutoff instead makes marginal early traders wait. Because the disclosed prices adjust endogenously, either ordering is possible.

The fixed-point condition leaves the seller with zero expected gain relative to the auction within each disclosed-state offer pool. The selected private full-menu equilibrium likewise gives zero rent to the seller state at the margin of each active price. Full disclosure therefore removes the extra rent seller L earns from a knockout price high enough to induce seller H to accept:

$$U_S^{\text{FD}} - U_S^D = -(1 - \pi)M_K \int_{\underline{b}}^{\bar{b}} [C_H(b) - C_L(b)] h_K(b) db \leq 0. \quad (\text{OA2.10})$$

Corollary OA2.2 (Ex-ante disclosure commitment). *Suppose that, before observing s , the seller chooses between the selected opaque equilibrium and commitment to the canonical disclosed-state assessment. If disclosure costs $\delta \geq 0$, the seller weakly prefers opacity. The preference is strict if $M_K > 0$ or $\delta > 0$.*

Proof. The disclosure commitment yields $U_S^{\text{FD}} - \delta$. Equation (OA2.10) gives

$$U_S^D - (U_S^{\text{FD}} - \delta) = \delta + (1 - \pi)M_K \int [C_H(b) - C_L(b)] h_K(b) db \geq 0,$$

with the stated strictness conditions. $\square$

For total surplus, couple the two regimes on the same realization and let A_j and O_j denote acceptance and offer under $j \in \{D, \text{FD}\}$. Since $O_{\text{FD}} = A_{\text{FD}}$,

$$W^{\text{FD}} - W^D = \kappa \Pr_D(O_D \setminus A_D) + \mathbb{E}[(A_{\text{FD}} - A_D)\{d_B + d_{S,s} - L_s(b) - \kappa\}]. \quad (\text{OA2.11})$$

The first term is the effort saved by avoiding rejected probes: under opacity the buyer sometimes incurs κ and learns through rejection that the seller has the high continuation state, whereas disclosure reveals that state before the buyer decides whether to make

an offer. The second term sums the welfare contributions of the added and removed early sales described by the cutoff changes above. An added early sale raises surplus exactly when $d_B + d_{S,s} > L_s(b) + \kappa$, because its timing gain then exceeds the foregone allocation through competition and the effort cost of the offer; removing an early sale has the opposite sign.

To isolate participation disclosure and the role of the prior, hold $F = U[0, 1]$, the common no-sale continuation value $v_0 = .4$, $(n_L, n_H) = (1, 3)$, $d_{S,L} = d_{S,H} = 0$, $\kappa = .02$, $\bar{d} = 1.2$, and $\lambda = 10$ fixed, and change only the probability π of the high-participation state. Table OA2.1 shows that raising π from .65 to .775 reverses the welfare effect of disclosure.

Table OA2.1: Opposite Welfare Effects of Participation-State Disclosure

Case	$\Pr_D(A)$	$\Pr_{FD}(A)$	$\Pr_D(O \setminus A)$	Saved rejections	Sale-set effect	$W^{FD} - W^D$
$\pi = .650$	.22533	.36103	.15322	.003064	-.001446	.001618
$\pi = .775$	.24259	.34158	.09736	.001947	-.002283	-.000336

Notes: “Saved rejections” is the first term in (OA2.11); “Sale-set effect” is the second. Entries are per object. Under disclosure every on-path offer is accepted, so disclosed offer and completion probabilities coincide.

The disclosed state-specific problems do not depend on the prior π , but the opaque screening menu does. At $\pi = .65$, avoiding rejected probes saves .003064, more than the .001446 loss from the changed early-sale set. Raising π to .775 reduces the opaque probing mass and hence the rejection saving to .001947. It also places more weight on the high-participation state, where early agreement displaces the stronger late competition; the total sale-set loss rises to .002283, of which .001607 arises in that state. Disclosure then lowers welfare by .000336 even though it eliminates every rejected attempt.

Proposition OA2.1 establishes a state-specific pooling equilibrium, with D1 survival under (OA2.7); accepted price menus may also carry further information about b . Corollary OA2.2 concerns ex-ante commitment to automatic disclosure of the seller state.

OA2.1.1 Interim Voluntary Hard Evidence

Now suppose the seller observes s and then either remains silent or pays δ to present a state-specific certificate. A certificate is feasible only for the matching state. Truthfulness is therefore imposed by the evidence technology, not generated by the seller’s reporting incentives. Silence selects the canonical opaque continuation, while a certificate selects the corresponding disclosed-state continuation.

Proposition OA2.3 (Costly voluntary hard evidence). *Conditional on these canonical continuations, for every $\delta > 0$ the unique pure reporting PBE has both seller states remain silent.*

Section OA2.2 proves the result and gives the zero-cost classification. At $\delta = 0$, silence pooling and weak selective disclosure can coexist. Conditional on the named continuations, every $\delta > 0$ selects silence among pure reporting profiles. The result applies to truthful state-specific certificates and a common seller reporting objective.

Both policy comparisons hold the buyer population, late participation, resolution benefits, seller timing costs, and terminal auction fixed. Endogenous entry, preparation, information acquisition, or search would add further equilibrium responses. Continued bargaining also changes rejection values and replaces the single attempt cost with the effort cost of all proposals (Section OA4.1). The welfare accounting must then use the resulting allocation, early agreements, and total costs.

OA2.2 Disclosure Proofs

Proof of Proposition OA2.1. Under no-early-offer behavior, $C_s(\bar{b})$ is the lowest price accepted under every posterior. Because its payoff increases in (b, d) , it is unprofitable for all buyers exactly when it is unprofitable at $(\bar{b}, \bar{d})$, yielding (OA2.5). If the inequality holds, assign lower off-path prices to $b = \bar{b}$; this D1-permitted belief induces rejection. If it fails, types near $(\bar{b}, \bar{d})$ gain from $C_s(\bar{b})$ under every posterior.

If the inequality fails, offer mass is positive at the upper endpoint of $[C_s(\underline{b}), C_s(\bar{b})]$. At $q = C_s(\underline{b})$, buyers in an upper-value set of positive measure submit an offer, and $C_s(b) > C_s(\underline{b})$ on a positive-measure subset because C_s is strictly increasing on a top interval. Hence $\hat{T}_s(C_s(\underline{b})) > C_s(\underline{b})$. At $q = C_s(\bar{b})$, the posterior mean lies below the upper endpoint, so the intermediate value theorem gives an interior fixed point. Its cutoff is optimal for the buyer, and Bayes' rule makes the seller indifferent at $\hat{q}_s$. Assign every other price to $b = \bar{b}$: undercuts are rejected, while accepted deviations cost at least $C_s(\bar{b}) > \hat{q}_s$, so none is profitable.

With public s , a surely rejected offer is dominated by waiting, proving (iii). For (iv), marginal type $(b, \widehat{D}_s(b; \hat{q}_s))$ needs acceptance probability $\kappa/(\kappa + \hat{q}_s - p)$ for an undercut $p < \hat{q}_s$. Types with cutoffs outside $[0, \bar{d}]$ need more and fail the strict-gain test, so the largest surviving continuation value is $C_s(b_s^0)$. Under (OA2.7), belief on this type rejects every undercut. Otherwise, some $p \in (C_s(b_s^0), \hat{q}_s)$ is accepted under every surviving posterior and benefits all marginal types. Under the sufficient condition $\widehat{D}_s(\bar{b}; \hat{q}_s) \geq 0$, positive offer mass also implies $\widehat{D}_s(\bar{b}; \hat{q}_s) < \bar{d}$. Hence $\bar{b}$ is a marginal value, $b_s^0 = \bar{b}$, and the interior fixed point satisfies $\hat{q}_s < C_s(\bar{b})$. $\square$

OA2.2.1 Voluntary Hard Evidence

Proof of Proposition OA2.3. Normalize each seller's terminal-auction payoff to zero. Under the selected opaque continuation after pooled silence, seller L earns $K_L^D \geq 0$ and seller H earns zero. Under a correctly matched state-specific continuation, either

seller earns zero before paying the certification cost. An H -seller mistaken for L can reject the induced offer and obtain zero; let $J_{LH} \geq 0$ denote an L -seller's payoff from being mistaken for H .

Pooling on silence is a PBE for every $\delta \geq 0$: relative to disclosure, the two seller states gain $K_L^D + \delta$ and δ , respectively. For $\delta > 0$, any profile in which H certifies gives that type $-\delta$, while silence followed by rejection guarantees zero. If only L certifies, it obtains $-\delta$ rather than the nonnegative imitation payoff J_{LH} . Full disclosure of the seller state likewise gives each state $-\delta$, while off-path silence and rejection guarantee zero. Hence no pure reporting profile involving disclosure is a PBE when $\delta > 0$, proving uniqueness within the named continuation class. $\square$

At $\delta = 0$, the inequalities become weak. Besides pooling on silence, H -only disclosure is a PBE. L -only disclosure is supportable if and only if $J_{LH} = 0$, and disclosure by both seller states is supportable when off-path silence is assigned to L and invokes the corresponding state-specific continuation. Reporting-stage D1 and the intuitive criterion do not select between silence pooling and H -only disclosure: each certificate has a singleton feasible-type set, and H 's deviation to silence is an on-path indifference. This appendix classification is conditional on the named offer continuations and does not establish refinement of the reporting profile with disclosure by both seller states.

OA3 Risk Aversion as a Source of Gains from Early Agreement

This section replaces or complements resolution benefits with insurance gains. Bilateral feasibility and the particular equilibrium constructions are distinguished below; risk aversion does not by itself establish every outcome of the baseline model.

OA3.1 Risk Aversion and Equilibrium Selection

Risk aversion changes the comparison between a certain early agreement and the payoff lottery generated by the announced round. Conditional on a known continuation state s , let the buyer's monetary risk premium ρ_B and the seller's monetary risk premium ρ_S equal expected terminal payoff minus its certainty equivalent, on the normalized payoff scale. A larger premium means that the party is willing to surrender more expected payoff to replace the lottery with certainty. A mutually acceptable early price exists exactly when

$$d_B + d_{S,s} + \rho_B + \rho_S \geq \Gamma_s(b) + \kappa. \quad (\text{OA3.1})$$

The left-hand side collects the buyer's resolution benefit, the seller's saved timing cost, and the parties' insurance values; these gains must cover the competition wedge $\Gamma_s(b)$ and

the offer cost κ . Equation (OA3.1) is a state-conditional bilateral-feasibility condition, not an equilibrium cutoff. In the signaling game the buyer does not observe s , and probing and knockout insure different compound lotteries. Their relevant risk premia are therefore action-specific and cannot generally be inserted into the baseline cutoffs as a common additive term.

A positive seller risk premium lowers the seller's minimum acceptable early price, but seller risk aversion also makes its acceptance threshold a nonlinear certainty equivalent of the posterior payoff lottery rather than the posterior mean in (2.7). We therefore establish the general feasibility effect for both parties and construct the signaling equilibrium using privately heterogeneous buyer risk aversion with a risk-neutral seller.

Seller-posted buyout models show that a certain early purchase can insure risk-averse buyers or sellers (Budish and Takeyama, 2001; Reynolds and Wooders, 2009; Mathews and Katzman, 2006). Equation (OA3.1) formalizes the insurance interpretation proposed by Flått et al. (2022); the baseline supplies heterogeneous benefits from early resolution as a second mechanism consistent with a higher conditional pre-emption price. In our buyer-proposed setting, the buyer's privately observed risk aversion determines who offers, while the offered price screens the seller's continuation state. This interaction produces the probing equilibrium below without buyer or seller urgency.

Proposition OA3.1 (Probing with buyer risk aversion). *Consider the late-participation-only specification with*

$$(n_L, n_H, \pi, \kappa) = (2, 7, .5, .01), \quad b \sim U[0, 1], \\ v_L = v_H = 0, \quad d_B = d_{S,L} = d_{S,H} = 0.$$

The seller and late bidders are risk neutral. The early buyer has constant absolute risk aversion (CARA) utility $u_\alpha(x) = -e^{-\alpha x}$ over monetary payoff and privately observes an independent coefficient of absolute risk aversion

$$\alpha \in \{\alpha_L, \alpha_H\}, \quad \Pr(\alpha = \alpha_L) = \Pr(\alpha = \alpha_H) = \frac{1}{2},$$

where $\alpha_L = .37$ and α_H is the unique root in the certified interval $[2.78802798023663, 2.78802798043663]$. There is a waiting-probing PBE at $q \simeq .655960$. Every α_L -buyer waits, while an α_H -buyer offers q if and only if $b \geq .811700$. Seller L accepts and seller H rejects. The probabilities of no early offer, a rejected attempt, and successful pre-emption are

$$(.905850, .047075, .047075).$$

The assessment survives criterion D1 over the complete rationalizable seller-plan set.

The auction exposes the early buyer to uncertainty about both whether it wins and how much it pays. A probing offer replaces that lottery with the certain payoff $b - q - \kappa$ in

state L , while preserving the auction after rejection in state H . The construction uses the resulting demand for partial insurance: the less-risk-averse class waits, whereas sufficiently high-value buyers in the more-risk-averse class probe. Within either class, the gain from probing rises with match value.

Criterion D1 attributes an unused lower price to the top-value type in the less-risk-averse class and an unused higher price to the top-value type in the more-risk-averse class. The corresponding seller responses make neither deviation profitable. Outward-rounded interval arithmetic certifies the price, match-value cutoff, risk-aversion coefficient, and the continuum inequalities used in this refinement argument. The construction extends to a nearby calibrated family and to two sufficiently narrow continuous risk-aversion bands, with the upper band's center adjusting to preserve seller indifference. CARA is tractable here because a certain cash shift factors out of exponential expected utility. Section OA3.2 gives the proof, the precise perturbation construction, and the bilateral-feasibility condition.

OA3.2 Risk Aversion

OA3.2.1 Bilateral Feasibility with Risk Premia

Condition on a known continuation state s . The early buyer's terminal-auction surplus is

$$X_s(b) := \max\{b - \max\{v_s, Y_{1N}\}, 0\},$$

and the seller's terminal monetary payoff is $P_s(b)$ from (2.2). Their expected payoffs are $\mathbb{E}_s[X_s(b)] = b - w_s(b)$ and $\mathbb{E}_s[P_s(b)] = R_s(b)$. Let CE_B and CE_S denote monetary certainty equivalents at the relevant initial wealth. Monetary payoffs use the normalization in the main paper, Section 2; subtracting d_B from the buyer's payoff also shifts its initial wealth by d_B , preserving its utility over outcomes. Suppressing dependence on (s, b) , define each monetary risk premium as expected net payoff minus its certainty equivalent:

$$\rho_B = \mathbb{E}_s[X_s(b)] - d_B - \text{CE}_B(X_s(b) - d_B), \quad (\text{OA3.2})$$

$$\rho_S = R_s(b) - d_{S,s} - \text{CE}_S(P_s(b) - d_{S,s}). \quad (\text{OA3.3})$$

Concavity implies $\rho_B, \rho_S \geq 0$. The buyer weakly prefers a certain early trade at price p when

$$b - p - \kappa \geq \text{CE}_B(X_s(b) - d_B) = b - w_s(b) - d_B - \rho_B,$$

so its maximum early price is $p_B^{\max} = w_s(b) + d_B + \rho_B - \kappa$. The seller weakly prefers early trade when $p \geq p_S^{\min}$, where

$$p_S^{\min} = \text{CE}_S(P_s(b) - d_{S,s}) = R_s(b) - d_{S,s} - \rho_S.$$

The interval $[p_S^{\min}, p_B^{\max}]$ is nonempty exactly when (OA3.1) holds. The risk premia, buyer resolution benefit, and saved seller timing cost increase what the parties can jointly devote to early agreement; the competition wedge and offer cost are what must be financed.

This calculation concerns bilateral feasibility conditional on s . In the equilibrium game, the buyer offers before observing the seller's state. The certainty equivalent of the resulting compound lottery is generally not the probability-weighted average of state-specific certainty equivalents. Probing removes auction risk only when seller L accepts, whereas knockout removes it in both states, so the relevant buyer risk premium is action-specific. Recovering the baseline cutoffs by adding risk premia to the direct gains from early resolution requires the same monetary risk premium to apply across states and acceptance profiles.

Risk aversion changes the value of the auction lottery but not the allocation rule: truthful bidding remains weakly dominant in the committed second-price benchmark for any increasing utility function. Thus the extension changes the buyer's ex-ante choice between waiting and offering, not its optimal terminal bid.

OA3.2.2 Buyer Risk Aversion and Equilibrium Sorting

For Proposition OA3.1, specialize to deterministic late-bidder counts $n \in \{2, 7\}$, zero no-sale value, and iid $U[0, 1]$ late values. Write $Y_{1n} \geq Y_{2n}$ for the two highest values among the n late buyers. For a buyer with CARA utility $u_\alpha(x) = -e^{-\alpha x}$, the coefficient $\alpha > 0$ measures absolute risk aversion. Its terminal-auction surplus against n late buyers is $X_n(b) = \max\{b - Y_{1n}, 0\}$. Define the exponential moment of that surplus and its value-normalized form by

$$M_n(b, \alpha) = \mathbb{E}[e^{-\alpha X_n(b)}], \quad Z_n(b, \alpha) = e^{\alpha b} M_n(b, \alpha) = \mathbb{E}[e^{\alpha \min\{b, Y_{1n}\}}].$$

Expected terminal utility is $-M_n(b, \alpha)$, so maximizing expected utility is equivalent to minimizing M_n . The normalization Z_n removes the common cash component $e^{-\alpha b}$ and simplifies the comparison of actions. For uniform late values,

$$M_n(b, \alpha) = 1 - b^n + n \int_0^b y^{n-1} e^{-\alpha(b-y)} dy.$$

The risk-neutral seller's expected terminal revenue conditional on the early buyer's value is

$$R_n(b) := \mathbb{E}[\max\{Y_{2n}, \min\{b, Y_{1n}\}\}] = \frac{n-1}{n+1} + b^n - \frac{n}{n+1} b^{n+1}.$$

Thus $R_L = R_2$ and $R_H = R_7$ in this late-participation-only specialization. With equal prior probability on the two states, waiting gives the exponential moment

$$W(b, \alpha) = \frac{1}{2} M_2(b, \alpha) + \frac{1}{2} M_7(b, \alpha).$$

A probe at price q is accepted in state L , rejected in state H , and incurs κ in either state. Its exponential moment is therefore

$$P(b, \alpha; q) = e^{\alpha\kappa} \left[\frac{1}{2} e^{-\alpha(b-q)} + \frac{1}{2} M_7(b, \alpha) \right].$$

The buyer prefers probing exactly when $P \leq W$. Because expected utility equals the negative of these moments, the following positive rescaling is the buyer's scaled expected-utility gain from probing:

$$\mathcal{F}(b, \alpha; q) = 2e^{\alpha b} \{W(b, \alpha) - P(b, \alpha; q)\} = Z_2(b, \alpha) - e^{\alpha(q+\kappa)} - (e^{\alpha\kappa} - 1)Z_7(b, \alpha).$$

Thus $\mathcal{F} \geq 0$ means probing. Its derivative with respect to match value is

$$\mathcal{F}_b = \alpha e^{\alpha b} \left[(1 - b^2) - (e^{\alpha\kappa} - 1)(1 - b^7) \right].$$

For both calibrated risk-aversion coefficients, $e^{\alpha\kappa} - 1 < 2/7$. Because $(1 - b^2)/(1 - b^7) \geq 2/7$, we have $\mathcal{F}_b > 0$ for $0 < b < 1$: within either risk-aversion class, a higher match value makes probing more attractive and there is at most one match-value cutoff.

Set $\alpha_L = .37$, and let β denote the match-value cutoff for the more-risk-averse class: an α_H -buyer probes exactly when $b \geq \beta$. We construct a parameter point and equilibrium by solving for (q, β, α_H) . The first equation makes the highest-match-value buyer in the α_L class indifferent, the second makes seller L indifferent at the selected probing price given the pool $b \geq \beta$, and the third makes the buyer with $(b, \alpha) = (\beta, \alpha_H)$ indifferent:

$$\mathcal{F}(1, \alpha_L; q) = 0, \tag{OA3.4}$$

$$q = \frac{1}{1 - \beta} \int_{\beta}^1 R_2(t) dt = \frac{3 + \beta + \beta^2 - \beta^3}{6}, \tag{OA3.5}$$

$$\mathcal{F}(\beta, \alpha_H; q) = 0. \tag{OA3.6}$$

Verified interval arithmetic, which expands every numerical bound to include possible rounding error, locates a solution at

$$\begin{aligned} q &= .655960451221777\dots, \\ \beta &= .811699930342779\dots, \\ \alpha_H &= 2.7880279803\dots \end{aligned}$$

Equation (OA3.4) and monotonicity imply that every α_L -type below $b = 1$ waits. Equations (OA3.5)–(OA3.6) imply that α_H -types probe exactly when $b \geq \beta$. Seller L is indifferent and accepts, whereas seller H 's rejection margin is .1954402.

For the D1 refinement, the two seller states have disjoint ranges of possible posterior continuation values:

$$R_2(b) \in [1/3, 2/3], \quad R_7(b) \in [3/4, 7/8].$$

The separation orders all posterior-rationalizable seller responses to an unused price: low prices are rejected in both states, intermediate prices may be accepted only in state L , and sufficiently high prices may be accepted in both states, with mixing possible only at seller indifference. For every potentially profitable price below q , D1 leaves the top-value type in the less-risk-averse class, $\theta_L^* = (1, \alpha_L)$, undominated. Under that belief both sellers reject because $R_2(1) = 2/3 > q$. For potentially profitable prices above q , the corresponding D1-undominated type is the top-value type in the more-risk-averse class, $\theta_H^* = (1, \alpha_H)$, and seller H rejects. The remaining price intervals offer no buyer type a strict gain. Thus the beliefs

$$\mu(\cdot | p) = \begin{cases} \delta_{\theta_L^*}, & p < q, \\ \delta_{\theta_H^*}, & p > q, \end{cases}$$

together with sequentially optimal seller responses, cover the complete price line. The certificate verifies these D1 gain comparisons for every buyer type and every unused price, including the points at which the seller's optimal response changes, as well as every strict seller-response margin. This proves Proposition OA3.1.

OA3.2.3 Calibrated Continuation and Continuous Risk-Aversion Bands

Let λ denote the probability of the more-risk-averse class. The equilibrium equations, with state and class probabilities restored, have a nonsingular Jacobian with respect to (q, β, α_H) . The implicit function theorem therefore gives a nearby calibrated family $(q, \beta, \alpha_H) = g(\pi, \kappa, \alpha_L, \lambda)$, preserving the sorting and strict seller and D1 margins. Here α_H adjusts jointly with the equilibrium price and cutoff. This result does not establish robustness to independent changes in all primitive coefficients, including α_H .

An explicit construction also replaces the two atoms by continuous bands. Hold the displayed $(\pi, \kappa, \alpha_L, q)$ fixed. For $\varepsilon > 0$, put half the probability uniformly on $[\alpha_L - \varepsilon, \alpha_L]$ and half uniformly on $[c(\varepsilon) - \varepsilon, c(\varepsilon) + \varepsilon]$. These distributions converge to the binary distribution as $\varepsilon \rightarrow 0$ and $c(\varepsilon) \rightarrow \alpha_H$. The lower band is one-sided because the top lower-class buyer is initially indifferent. The certified inequality $\mathcal{F}_\alpha(1, \alpha_L; q) > .01166$, together with $\mathcal{F}(1, \alpha_L; q) = 0$, makes all lower-band buyers wait for sufficiently small ε , apart from the indifferent endpoint. The strict condition $e^{\alpha\kappa} - 1 < 2/7$ preserves monotonicity in $b < 1$ throughout both bands.

For α near α_H , define the smooth cutoff $\beta(\alpha)$ by $\mathcal{F}(\beta(\alpha), \alpha; q) = 0$. Let

$$H(\alpha) = \int_{\beta(\alpha)}^1 [R_2(b) - q] db, \quad \mathcal{G}(c, \varepsilon) = \frac{1}{2} \int_{-1}^1 H(c + \varepsilon t) dt.$$

Seller L 's zero-rent condition is $\mathcal{G}(c, \varepsilon) = 0$. At $(c, \varepsilon) = (\alpha_H, 0)$, it holds by (OA3.5), and

$$\mathcal{G}_c(\alpha_H, 0) = -[R_2(\beta) - q]\beta'(\alpha_H), \quad \beta'(\alpha_H) = -\frac{\mathcal{F}_\alpha}{\mathcal{F}_b} < 0.$$

Interval arithmetic encloses $\mathcal{G}_c$ in $-.0005138999 \pm 4.42 \times 10^{-11}$, strictly below zero. The implicit function theorem therefore supplies $c(\varepsilon)$ for all sufficiently small positive ε , with q unchanged. The offer posterior weights (b, α) by $\mathbf{1}\{b \geq \beta(\alpha)\}$ on the upper band. Its normalization is positive; $\mathcal{G} = 0$ gives seller L 's indifference, while seller H 's strict rejection margin persists. The positive waiting, rejection, and agreement probabilities converge to those of the binary assessment.

To check D1 uniformly over unused prices, consider first $p \in [1/3, q)$, where only seller L can accept. Write $S_n = 1 - b^n$. A waiting buyer's threshold acceptance probability has derivative in b with the sign of

$$N_\alpha(b, p) = (S_2 + S_7)(Z_2 - e^{\alpha p}) - S_2(Z_2 + Z_7).$$

Throughout the lower band, $N_\alpha/(S_2 + S_7) \leq e^{\alpha/3}[(5/9)e^{2\alpha/3} - 1] < 0$ for $b < 1$, so every within-coefficient minimizer has $b = 1$. Near the upper band, the certified bound $N_\alpha(b, 1/3) < -.173$ for waiting values persists; the prober's threshold increases in b , so the upper-band minimizers are the cutoff types. To compare these with $(1, \alpha_L)$, put $\delta = q - p$, $\varphi(z) = (1 - e^{-z})/z$, with $\varphi(0) = 1$, and

$$A(b, \alpha, \delta) = \frac{\alpha e^{\alpha q}}{Z_2(b, \alpha) - e^{\alpha q}} \varphi(\alpha \delta).$$

For indifferent types the threshold is $1/[1 + \delta A]$. The binary certificate gives $A(1, \alpha_L, \delta) - A(\beta, \alpha_H, \delta) > 7.82$ uniformly on $0 \leq \delta \leq q - 1/3$. Continuity on this compact interval preserves the strict comparison with every nearby upper-band cutoff, including as $p \uparrow q$. D1 therefore removes all upper-band types and all lower-band types with $b < 1$ whenever a strict gain is possible. Every surviving belief has $b = 1$, under which both sellers reject since $p < q < R_2(1)$.

For $p \in [3/4, 7/8]$, seller L accepts and only seller H 's acceptance probability can vary. Among hypothetical probers the threshold is minimized at $b = 1$, where it is

$$x_T(\alpha; p) = \frac{e^{\alpha p} - e^{\alpha q}}{Z_7(1, \alpha) - e^{\alpha p}}.$$

A waiter has a weakly higher threshold. On the positive-gain region, x_T strictly decreases in α : after dividing by $\alpha e^{\alpha p}$, its numerator decreases and its denominator increases in α . Thus the top-value buyer with coefficient $c(\varepsilon) + \varepsilon$ survives D1, and seller H rejects. The highest potentially profitable price remains strictly below $7/8$, since the binary bound is $p^\dagger < .792$. The remaining price intervals either admit no acceptance or reproduce the probing response at a higher price. These beliefs deter every deviation. Hence the two-band assessment is a D1-compatible PBE for all sufficiently small $\varepsilon > 0$. The argument gives no explicit positive band width and retains a gap between the bands; it does not establish the result for an arbitrary connected distribution of risk aversion.

OA4 Alternative Bargaining and Sale Protocols

The English auction in the main paper is one tractable continuation mechanism. The following protocols ask how selected incentives change when negotiations continue before the auction, a counteroffer follows it, or a private offer arrives during bidding. These are separate variations, not prerequisites for the main model. The last analysis holds the bidding history fixed and remains tentative; it is not a full dynamic equilibrium characterization.

OA4.1 Pre-emption-Stage Bargaining

The baseline makes the buyer's early offer final: rejection leads directly to the planned auction. Suppose instead that either party may make another binding proposal, at nonnegative proposal costs c_B and c_S , while either party may always end bargaining and activate the unchanged auction. If continued bargaining is valuable to urgent buyers, patient high-value buyers may imitate them and share the resulting information rent. Renegotiation can therefore change both whether early agreement occurs and which buyer types initiate. Formally, if $\Omega_s(p) \geq 0$ is seller type s 's incremental value of retaining this option after rejecting p , its effective rejection value is

$$\mathcal{R}_s(p) = \mathcal{C}_s(p) + \Omega_s(p), \tag{OA4.1}$$

$$\mathcal{R}_L(p) \leq p < \mathcal{R}_H(p). \tag{OA4.2}$$

The second line is necessary for the initial offer to separate the seller states. Full equilibrium additionally requires jointly sequentially rational proposals, exit decisions, posteriors, and continuation play, as in the bargaining literature with two-sided incomplete information (Cramton, 1984; Rubinstein, 1985; Grossman and Perry, 1986; Chatterjee and Samuelson, 1987; Fudenberg et al., 1987; Chatterjee and Samuelson, 1988; Cramton, 1992; Watson, 1998); Ausubel et al. (2002) survey this literature.

Proposition OA4.1 (Urgency and pooling under continued pre-emption-stage bargaining). *Consider the unrestricted alternating-offer extension above, with an initial attempt cost $\kappa > 0$.*

- (i) *Under a common prior and risk neutrality, if $d_B = d_{S,s} = 0$ almost surely, the auction remains available after every proposal, and perpetual bargaining cannot raise joint material payoff above the auction, every PBE has zero early-attempt probability.*
- (ii) *In the late-participation-only specialization with common $U[0, 1]$ private values, there are strictly positive proposal costs and an atom-continuous buyer-urgency distribution for which an unrestricted-game PBE has no early offer, an attempt followed by the auction, and early agreement, each with positive probability. A positive mass of buyers with $d_B = 0$ initiates. No round cap is imposed, although the selected path ends after at most three proposals.*

The proof below specifies strategies and beliefs after every finite history. Urgent buyers create gains from early resolution, while patient high-value buyers pool with them and earn information rents through the revision path. Commitment to proceed directly to the bidding round after rejection removes this rent. Pooling can reduce allocative efficiency, but urgent types obtain an intrinsic benefit from early resolution, leaving the overall welfare effect ambiguous.

OA4.1.1 Proof of Proposition OA4.1

For part (ii), specialize to $v_L = v_H = 0$, deterministic late counts $(n_L, n_H) = (2, 5)$, $d_S = 0$, and $\pi = \Pr(s = H) = 3/10$. Let b and the i.i.d. sequence $(Y_i)_{i \geq 1}$ be mutually independent and uniform on $[0, 1]$; the first n_s late buyers participate in state s . Independently draw buyer urgency from

$$G_\eta = \eta \delta_0 + (1 - \eta)U[0, 1]. \quad (\text{OA4.3})$$

Set

$$\kappa = \frac{1}{2000}, \quad c_B = \frac{1}{1000}, \quad c_S = \frac{1}{500}, \quad \beta = \frac{9}{10},$$

where β is the patient buyer's initiation cutoff, and use initial offer $q = 1/2$, seller counteroffer $r = 3797/5000$, and buyer revision $p = 573/875$. The construction applies for every

$$0 \leq \eta < \eta^\dagger, \quad \eta^\dagger := \frac{117204341}{120480341}. \quad (\text{OA4.4})$$

Proof. For part (i), let I be the event that the buyer makes a binding initial offer. Fix the realized private types and bargaining history while integrating over terminal-auction randomness. Let $K_B + K_S \geq \kappa$ be the total proposal costs on an attempted history, and

define gains relative to the planned auction by $g_B = U_B - u_s(b)$ and $g_S = U_S - C_s(b)$. If bargaining fails, then

$$g_B + g_S = -(K_B + K_S).$$

If agreement occurs, the price cancels between the parties and

$$g_B + g_S = b - u_s(b) - C_s(b) - (K_B + K_S) = -\Gamma_s(b) - (K_B + K_S).$$

The assumed completion gives the same upper bound on an infinite history. Hence every attempted history satisfies $g_B + g_S \leq -\kappa$.

At the buyer's initial information set, waiting gives its conditional expected auction payoff. Initial optimality therefore implies $\mathbb{E}[g_B \mathbf{1}_I] \geq 0$. At the seller's first response, condition on its full information, including its type, the observed proposal, and the preceding contact history. The seller can invoke the unchanged auction before incurring any voluntary continuation cost. Under regular conditional beliefs, defined almost everywhere and updated by Bayes' rule on reached histories, seller optimality and iterated expectations imply $\mathbb{E}[g_S \mathbf{1}_I] \geq 0$. Because both constraints use the common prior,

$$0 \leq \mathbb{E}[(g_B + g_S) \mathbf{1}_I] \leq -\kappa \Pr(I),$$

so $\Pr(I) = 0$.

For part (ii), use the mutually independent construction above. Here $1 - \pi = 7/10$, and Equation (OA5.2) gives $w_s = w_{n_s,0}$ and $C_s = R_{n_s,0}$. Since $w_L(\beta) = 657/1000$, the stated prices satisfy

$$p = w_L(\beta) - \frac{c_B + \kappa}{1 - \pi} = \frac{573}{875}, \quad r = (1 - \pi)w_L(\beta) - \kappa + \pi = \frac{3797}{5000}, \quad q = \frac{1}{2} < p < r.$$

After the seller counteroffer r , termination, revision to p , and acceptance of r yield, respectively,

$$0, \quad P(b, d) = (1 - \pi)[w_L(b) + d - p] - c_B, \quad E(b, d) = (1 - \pi)w_L(b) + \pi w_H(b) + d - r.$$

Define

$$\begin{aligned} a(b) &= p + \frac{c_B}{1 - \pi} - w_L(b), & x(b) &= a(b) + \frac{\kappa}{1 - \pi} = w_L(\beta) - w_L(b), \\ y(b) &= \frac{r - (1 - \pi)p - c_B - \pi w_H(b)}{\pi} = 1 - w_H(b). \end{aligned}$$

The buyer waits initially when $d < \max\{x(b), 0\}$ and offers q otherwise, with initiation prescribed at the null tie $(\beta, 0)$. Every type that reaches r , including types arriving only through a tremble, follows the complete strategy

$$\begin{cases} \text{terminate,} & d < a(b), \\ \text{revise to } p, & a(b) \leq d < y(b), \\ \text{accept } r, & d \geq y(b). \end{cases}$$

Indeed, a equates revision and termination, while y equates acceptance and revision. Moreover,

$$\frac{d}{db}[y(b) - x(b)] = b^5 - b^2 < 0, \quad y(\beta) > x(\beta) = 0.$$

Thus $x(b) < y(b)$ below β , while $a(b) < x(b) < 0 < y(b)$ above it. Every on-path initiator has $d \geq \max\{x(b), 0\} > a(b)$, so termination is off path. A patient type strictly waits below β and strictly initiates and revises above it.

For the continuous urgency component, waiting, revision, acceptance of r , and initiation have conditional weights

$$\begin{aligned} e_0(b) &= \max\{x(b), 0\}, & m_0(b) &= y(b) - \max\{x(b), 0\}, \\ h_0(b) &= 1 - y(b), & z_0(b) &= 1 - \max\{x(b), 0\}. \end{aligned}$$

The patient atom waits below β , revises above it, and never accepts r . Hence

$$\begin{aligned} e_\eta(b) &= (1 - \eta)e_0(b) + \eta \mathbf{1}\{b < \beta\}, & m_\eta(b) &= (1 - \eta)m_0(b) + \eta \mathbf{1}\{b \geq \beta\}, \\ h_\eta(b) &= (1 - \eta)h_0(b), & z_\eta(b) &= (1 - \eta)z_0(b) + \eta \mathbf{1}\{b \geq \beta\}. \end{aligned}$$

Integrating over $b \sim U[0, 1]$ gives

$$\begin{aligned} Z_e &= \frac{9639 + 26361\eta}{40000}, & Z_p &= \frac{237581 - 153581\eta}{840000}, \\ Z_h &= (1 - \eta)\frac{10}{21}, & Z_q &= \frac{30361 - 26361\eta}{40000}, \end{aligned}$$

where $Z_e + Z_p + Z_h = 1$ and $Z_p + Z_h = Z_q$. Bayes' rule makes the post- q density z_η/Z_q and the post- p density m_η/Z_p . For $s \in \{L, H\}$, define the corresponding unnormalized auction values

$$N_{sq} = \int_0^1 z_\eta(b) C_s(b) db, \quad N_{sp} = \int_0^1 m_\eta(b) C_s(b) db.$$

After q , both seller states counter at r ; after p , seller L accepts and seller H invokes the auction. Before subtracting c_S , the counteroffer payoffs are $V_L = pZ_p + rZ_h$ and $V_H = N_{Hp} + rZ_h$. The complete on-path seller inequalities are

$$V_s > (q + c_S)Z_q, \quad V_s > c_S Z_q + N_{sq} \quad (s = L, H),$$

$$pZ_p > N_{Lp}, \quad N_{Hp} > pZ_p.$$

Exact integration gives the six unnormalized margins

$L : r \text{ versus } q$	$(97493167 - 88505167\eta)/588000000$
$L : r \text{ versus auction}$	$(209471659 - 211036159\eta)/1470000000$
$H : r \text{ versus } q$	$(734314608737 - 601422352487\eta)/4095000000000$
$H : r \text{ versus auction}$	$(117204341 - 120480341\eta)/16380000000$
$L : p \text{ versus auction}$	$(266643493 - 271217293\eta)/5292000000$
$H : \text{auction versus } p$	$(387410369909 + 104670423841\eta)/28665000000000$

The fourth margin is the first to reach zero, at $\eta = \eta^\dagger$ in (OA4.4); every other margin remains strictly positive there. The stated seller actions are therefore strict throughout the interval in the proposition. At equality the same path is a weak PBE if seller H counters at r when indifferent; above it, H strictly prefers the auction after q .

It remains to complete later histories. Write $\bar{C}_{sq} = N_{sq}/Z_q$ and $\bar{C}_{sp} = N_{sp}/Z_p$. For a preceding on-path buyer offer $j \in \{q, p\}$, let

$$\varepsilon = \frac{c_S - c_B}{2} = \frac{1}{2000}, \quad \tau_j = \bar{C}_{Lj} + \varepsilon, \quad T_j = \tau_j + c_B.$$

After an unexpected seller price ζ , the buyer believes state L . Writing $v = w_L(b) + d$, a type with $v > T_j$ accepts when $\zeta \leq T_j$ and offers τ_j otherwise; a type with $v \leq T_j$ accepts when $\zeta \leq v$ and terminates otherwise. These rules compare $v - \zeta$, $v - T_j$, and zero and therefore define a best response for every actual buyer type.

Every type in either on-path posterior has $v \geq w_L(\beta) > T_j$. Following a high seller price, all such types pool on τ_j , and the selected assessment retains the preceding posterior. Seller L strictly accepts by ε , while seller H strictly invokes the auction. The unnormalized versions of

$$w_L(\beta) - T_j, \quad \bar{C}_{Hj} - \tau_j, \quad C_L(1) - \tau_j$$

are affine in η and positive at both zero and $\eta^\dagger$; at the latter endpoint their smallest values across $j \in \{q, p\}$ are, respectively, 0.005684921, 0.167132113, and 0.016351588. Moreover, a seller price at or below T_j yields at most

$$T_j - c_S = \bar{C}_{Lj} + \varepsilon + c_B - c_S < \bar{C}_{Lj}.$$

Above T_j , seller L receives $\tau_j - c_S < \bar{C}_{Lj}$, while seller H invokes the auction and receives $\bar{C}_{Hj} - c_S < \bar{C}_{Hj}$. No further seller counteroffer therefore beats the posterior auction value.

At every nonspecial buyer price, use the off-path belief $\delta_{(1,1)}$. The seller's acceptance thresholds are then $C_L(1) = 2/3$ and $C_H(1) = 5/6$. Apply the same response rule to every actual buyer type with

$$\tau_* = C_L(1), \quad T_* = C_L(1) + c_B.$$

The believed type can use this tail with payoff margin 999/1000. Seller L accepts τ_* while indifferent to the auction, seller H strictly auctions, and, because $c_S > c_B$, every further seller counteroffer is strictly worse. The four binding global buyer-price comparisons are

$$(1 - \pi)[C_L(1) - p] - c_B = \frac{109}{15000}, \quad C_H(1) - r = \frac{1109}{15000},$$

$$(1 - \pi)[C_L(1) - p] = \frac{31}{3750}, \quad C_H(1) + c_B - r = \frac{281}{3750},$$

and are strictly positive. They cover initial and revised offers accepted only in state L or in both seller states; rejected prices are dominated by not making the proposal. Together with the recursive tail, these inequalities exhaust buyer deviations at every continuation history.

For a Bayes-limit check, preserve atoms on the prescribed q - and p -history tails and add vanishing full-support price densities. At a nonspecial buyer offer, choose type-dependent trembles whose likelihood concentrates on $\{(b, d) : b > 1 - \delta_n, d > 1 - \delta_n\}$, $\delta_n \downarrow 0$, while every tremble probability vanishes. The seller's posterior then converges to $\delta_{(1,1)}$. At every finite history h followed by an unexpected seller price ζ , choose positive full-support subdensities $\lambda_{L,h}^n(\zeta)$ and $\lambda_{H,h}^n(\zeta)$ with vanishing total mass and relative rates such that the full L/H likelihood ratio, including the likelihood of h , diverges pointwise in ζ . They depend on the state and public history but not on the buyer's hidden type, and therefore leave the seller's conditional posterior over buyer types unchanged. Apply the construction recursively after every longer finite history. The buyer's posterior then converges to state L after every unexpected seller price. This supplies one atom-preserving, full-support regularization delivering the stated posteriors; it does not establish sequential equilibrium or dynamic D1 refinement.

Finally, for $\eta > 0$, patient buyers initiate with mass $\eta/10$. Their failure and agreement masses are $3\eta/100$ and $7\eta/100$. The total outcome probabilities are

$$\Pr(\text{no early offer}) = \frac{9639 + 26361\eta}{40000},$$

$$\Pr(\text{attempt then auction}) = \frac{237581 - 153581\eta}{2800000},$$

$$\Pr(\text{early agreement}) = \frac{1887689 - 1691689\eta}{2800000}.$$

All three are positive throughout (OA4.4). The buyer strategies, seller inequalities, recursive completion, and Bayes' rule thus define the claimed PBE, completing part (ii). $\square$

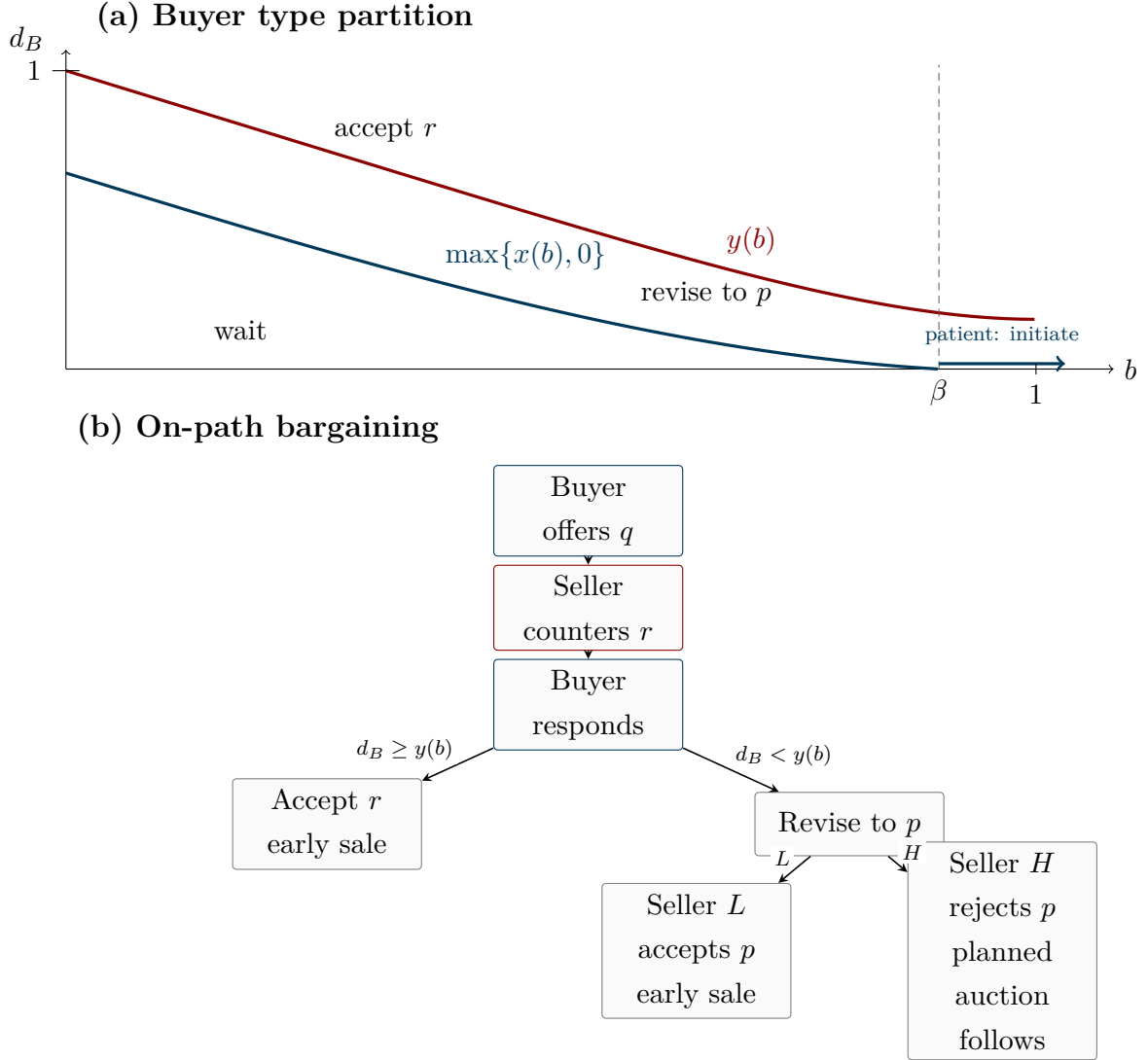


Figure OA4.1: Buyer type partition and on-path proposals in the repeated-bargaining PBE of Proposition OA4.1(ii).

OA4.2 Post-Bidding Counteroffers

A separate commitment concern arises after the bidding round. We study a continuous public English clock followed by one possible seller counteroffer. As a benchmark, the terminal decision cannot condition on the earlier pre-emption action. The seller observes

the clock history and the standing price x , set by the last losing dropout, but not the survivor's unsubmitted exit value. It may accept x , or release the survivor from that bid and make one final counteroffer $r > x$. Rejection of r ends without sale and gives the seller the public fallback v ; the survivor cannot revise and losing bidders cannot return (Koska and Stähler, 2021). We assume that the terminal seller has no action-linked selection information and therefore treats the survivor as an otherwise unselected iid draw from F . Conditional on x , the survivor's value then has distribution $F(\cdot \mid \cdot \geq x)$. If

$$r = \rho(v) \in \arg \max_{p \geq v} (p - v)[1 - F(p)]$$

is the unique interior maximizer and $(p - v)[1 - F(p)]$ is weakly decreasing for $p \geq r$, the seller counters at r when $x < r$ and accepts x otherwise, and truthful clock exit remains a weak best response. Thus terminal discretion creates an endogenous, unannounced threshold and can inefficiently exclude willing buyers. The no-action-linkage calculation below derives the survivor's conditional distribution and the resulting continuation objects.

Suppose instead that the seller can identify whether the survivor is the early buyer and, if so, whether that buyer waited or probed. The seller may then condition its counteroffer on that action.

Proposition OA4.2 (Post-bidding counteroffer with action linkage). *Let $F = U[0, 1]$, $(n_L, n_H) = (2, 3)$, $\pi = 1/2$, $v = .4$, $d_{S,L} = d_{S,H} = .01$, $\kappa = .01$, $\lambda = 10$, and $\bar{d} = .15$. There exists a waiting-probing PBE with*

$$(q, r_P, r_W) \simeq (.738406, .780957, .661140),$$

where r_P and r_W are the action-specific counteroffer thresholds after probing and waiting. Its outcome probabilities are

$$(\Pr(\text{no early offer}), \Pr(\text{rejection}), \Pr(\text{success})) \simeq (.865398, .067301, .067301).$$

Every on-path initiator has positive urgency. The assessment survives criterion D1 in the stated one-counteroffer contingent-plan representation, over the complete set of posterior-rationalizable early responses and sequentially optimal one-counteroffer rules, including irregular posterior-revenue rules.

The action-linkage analysis below gives the fixed-point system and refinement proof. An outward-rounded interval certificate establishes the root, optimality of both on-path counteroffers, and the inequalities used in the D1 argument. Hence screening and all three early outcomes can survive even when the seller identifies the winner's early action and tailors its terminal counteroffer accordingly.

OA4.2.1 No Action Linkage

When the terminal decision cannot condition on the earlier action, let $X_{(m)} \geq X_{(m-1)}$ denote the two highest values among the $m \geq 2$ active bidders. Conditional on the final English-clock price $X_{(m-1)} = x$, the survivor's value $X_{(m)}$ has density

$$f_{X_{(m)}|X_{(m-1)}=x,m}(y) = \frac{f(y)}{1 - F(x)} \mathbf{1}\{y \geq x\}.$$

This density is independent of m . For $x < \bar{b}$, a terminal price $p \geq \max\{x, v\}$ gives expected seller payoff $v + (p - v)[1 - F(p)]/[1 - F(x)]$. Thus, for a general distribution, the seller chooses

$$\sigma(x) \in \arg \max_{p \geq \max\{x, v\}} (p - v)[1 - F(p)].$$

Under the post-peak monotonicity condition stated above, r maximizes this expression when $x < r$, and x is a maximizer when $x \geq r$. Choosing acceptance in the latter case gives $\sigma(x) = \max\{x, r\}$. Uniqueness of the unconstrained maximizer alone does not imply this rule: revenue could rise again beyond a standing price above r . The continuation formulas below apply to the threshold rule; the constrained maximization remains valid without the shape restriction.

Let $Y_{1,n} \geq Y_{2,n}$ be the two highest values among n late buyers, and write $Y_{1s} = Y_{1,n_s}$. Define

$$R_F(n, b; r) := \mathbb{E}[\max\{r, Y_{2,n}, \min\{b, Y_{1,n}\}\}],$$

the seller's gross expected payoff when both the committed threshold and the no-sale payoff equal r . With counteroffer threshold r , the early buyer's willingness and the seller's continuation objects are

$$w_s^{CO}(b) = b - \mathbb{E}[\max\{b - \max\{r, Y_{1s}\}, 0\}],$$

$$R_s^{CO}(b) = R_F(n_s, b; r) - (r - v)F(r)^{n_s} \mathbf{1}\{b < r\}, \quad C_s^{CO}(b) = R_s^{CO}(b) - d_s. \quad (\text{OA4.5})$$

OA4.2.2 Action Linkage

Suppose instead that the terminal seller can identify an early-buyer survivor and remember whether that buyer waited or probed. If $\alpha_a(b)$ is the probability of action $a \in \{W, P\}$ at value b , define the unnormalized selected tail

$$T_a(z) = \int_z^1 \alpha_a(t) dF(t).$$

After the last losing bidder drops out at x , the action-specific terminal price satisfies

$$\sigma_a(x) \in \arg \max_{z \geq \max\{x, v\}} (z - v)T_a(z).$$

The normalizing probability $T_a(x)$ cancels from this maximization. Thus the early action affects both the terminal counteroffer and the buyer's payoff after a rejected early offer.

At the uniform primitives of Proposition OA4.2, let $Y_{1,n} \geq Y_{2,n}$ be the two highest values among n late buyers and let $\rho = (1 + v)/2$. For a sequential terminal-price rule σ with $\sigma(x) \geq \max\{x, v\}$, define

$$S_n^\sigma(b) = \mathbb{E}[\max\{b - \sigma(Y_{1,n}), 0\}].$$

The seller's corresponding gross terminal payoff conditional on b is $R_n^\sigma(b) = \mathbb{E}[P_n^\sigma(b)]$, where

$$\begin{aligned} P_n^\sigma(b) = & \mathbf{1}\{Y_{1,n} \leq b\} [\mathbf{1}\{b \geq \sigma(Y_{1,n})\} \sigma(Y_{1,n}) + \mathbf{1}\{b < \sigma(Y_{1,n})\} v] \\ & + \mathbf{1}\{Y_{1,n} > b\} [\mathbf{1}\{Y_{1,n} \geq \rho\} \max\{\rho, b, Y_{2,n}\} + \mathbf{1}\{Y_{1,n} < \rho\} v]. \end{aligned}$$

For the scalar rule $\sigma_r(x) = \max\{x, r\}$, write $S_n(b; r) = S_n^{\sigma_r}(b)$, $R_n(b; r) = R_n^{\sigma_r}(b)$, and

$$S_n(b; r) = \frac{\max\{b^{n+1} - r^{n+1}, 0\}}{n+1}, \quad w_s(b; r) = b - S_{n_s}(b; r), \quad C_s(b; r) = R_{n_s}(b; r) - d_S.$$

Thus r changes the counteroffer faced by an early-buyer winner; a late winner still faces the common threshold ρ .

For the waiting-probing assessment, let r_P and r_W denote the selected terminal thresholds after probing and waiting. Let $\tau = (b, d)$ denote a buyer type and set $A = 1 - \pi$. Its on-path waiting and probing payoffs are

$$\begin{aligned} U_W(b, d) &= AS_{n_L}(b; r_W) + \pi S_{n_H}(b; r_W) - d, & U^*(b, d) &= \max\{U_W(b, d), U_P(b, d)\}. \\ U_P(b, d) &= A(b - q) + \pi\{S_{n_H}(b; r_P) - d\} - \kappa, \end{aligned}$$

We also write $U^*(\tau) = U^*(b, d)$. The wait-probe urgency cutoff and probing probability are

$$\begin{aligned} D(b) &= q - w_L(b; r_W) + \frac{\kappa - \pi\{w_H(b; r_W) - w_H(b; r_P)\}}{A}, \\ \alpha_P(b) &= 1 - G(D(b)). \end{aligned}$$

The full-linkage fixed point solves

$$\begin{aligned} q &= \frac{\int \alpha_P(b) C_L(b; r_P) dF(b)}{\int \alpha_P(b) dF(b)}, \\ r_P &\in \arg \max_{r \geq v} (r - v) \int_r^1 \alpha_P(b) dF(b), \\ r_W &\in \arg \max_{r \geq v} (r - v) \int_r^1 [1 - \alpha_P(b)] dF(b). \end{aligned}$$

At the primitives in Proposition OA4.2, an outward-rounded ball-arithmetic calculation applies the Krawczyk theorem to the system above and proves a root in the box centered at

$$(q, r_P, r_W) = (.738406414005472, .780957223173591, .661140319939824)$$

with radius 10^{-10} in each coordinate. The Krawczyk image has coordinate radii below 5.8×10^{-17} , so the root is unique within this box. Interval differentiation also makes each selected posterior-revenue derivative strictly decreasing on its support. The two root coordinates are therefore the global maxima of their respective posterior-revenue curves. Hence $\sigma_a(x) = \max\{x, r_a\}$ is sequentially optimal after every on-path clock dropout.

After an unused price p , let ν be the marginal posterior over b , let $T_\nu(z) = \nu([z, 1])$, and let a receiver plan be $\alpha = (\sigma, a_L, a_H)$, where

$$\sigma(x) \in \arg \max_{z \geq \max\{x, v\}} (z - v) T_\nu(z)$$

and a_s is the probability that seller state s accepts the early price. Define

$$K_n(\nu, \sigma) = \int R_n^\sigma(b) d\nu(b) - d_S.$$

Sequential rationality requires $a_s = 1$ if $p > K_{n_s}(\nu, \sigma)$, $a_s = 0$ if $p < K_{n_s}(\nu, \sigma)$, and permits mixing at equality. Type $\tau = (b, d)$'s deviation payoff is

$$\begin{aligned} u_\tau(p; \alpha) &= A\{a_L(b - p) + (1 - a_L)[S_{n_L}^\sigma(b) - d]\} \\ &\quad + \pi\{a_H(b - p) + (1 - a_H)[S_{n_H}^\sigma(b) - d]\} - \kappa. \end{aligned}$$

A scalar-reserve witness uses $\sigma = \sigma_r$.

It remains to verify off-path incentives when the same posterior determines early acceptance and the subsequent terminal rule. Let

$$\tau_M = (1, D(1)), \quad D(1) \simeq .040637366, \quad \tau_U = (1, \bar{d}),$$

and let

$$p^\dagger = 1 - \kappa - U^*(1, \bar{d}) \simeq .865699572.$$

Equal-revenue posterior tails and mixing at seller indifference generate scalar-reserve witness plans. In the following table, (a_L, a_H) are early-acceptance probabilities and x is the mixing probability of the indifferent seller state:

unused-price interval	terminal threshold	(a_L, a_H)	undominated type
$p \leq .644733060$	.632096261	$(0, 0)$	τ_M
$p \in [.644733060, .727669887]$	r_W	$(x, 0)$	τ_M
$p \in [.727669887, q)$	.678	$(x, 0)$	τ_M
$p \in (q, .824480449]$	r_P	$(1, x)$	τ_U
$p \in [.80, p^\dagger]$	.85	$(1, x)$	τ_U .

In each row $x \in [0, 1]$ makes the displayed type indifferent, while no other buyer type gains strictly. The intervals overlap and cover every unused price at or below $p^\dagger$. Consequently, enlarging the receiver's plan set cannot D1-dominate the displayed type: its witness plan would have to belong to a proposed dominator's strict-gain set, but it belongs to no type's strict-gain set.

For completeness, consider an arbitrary posterior ν as above, possibly with several posterior-revenue peaks. Define

$$\Phi_\nu(z) = (z - v)T_\nu(z), \quad r_\nu = \min \arg \max_z \Phi_\nu(z), \quad M_\nu(x) = \max_{z \geq \max\{x, v\}} \Phi_\nu(z).$$

Every sequentially optimal counteroffer is at least $\max\{x, r_\nu\}$, and therefore

$$S_n^\sigma(b) \leq S_n(b; r_\nu).$$

Let Λ_n be gross seller revenue when the early buyer's value is zero:

$$\Lambda_n = v\rho^n + n(1 - \rho)\rho^n + (n - 1)(1 - \rho^n) - \frac{n(n - 1)}{n + 1}(1 - \rho^{n+1}).$$

Conditioning on the highest late value and integrating the late-winner term by parts gives

$$\begin{aligned} K_n(\nu, \sigma) + d_S &= \Lambda_n + \int_0^1 nx^{n-1}M_\nu(x) dx \\ &\quad + \int_\rho^1 nx^{n-1}(1 + v - 2x)T_\nu(x) dx. \end{aligned}$$

The final integrand is nonpositive, while $M_\nu(x) \leq (r_\nu - v)T_\nu(r_\nu) \leq r_\nu - v$. Hence

$$K_n(\nu, \sigma) \leq \Lambda_n + r_\nu - v - d_S$$

for every posterior-optimal one-counteroffer rule, without a regularity or scalar-reserve restriction.

Here $(\Lambda_{n_L}, \Lambda_{n_H}) = (.562, .62005)$. At $p = p^\dagger$, rejection by L requires $r_\nu \geq p^\dagger - .152 \simeq .713699572$, while rejection by H requires $r_\nu \geq p^\dagger - .21005 \simeq .655649572$. Combining these floors with the buyer-surplus bound gives the following rigorous upper bounds:

(a_L, a_H)	$(1, 1)$	$(1, 0)$	$(0, 1)$	$(0, 0)$
$\sup_\tau \{u_\tau(p^\dagger; \alpha) - U^*(\tau)\}$	0	$< -.036828520$	$< -.032651821$	$< -.008706110$.

Higher unused prices raise the relevant threshold floor and make accepted trade more expensive, so no type weakly gains above $p^\dagger$.

Finally, assign belief δ_{τ_M} below q and δ_{τ_U} above q through $p^\dagger$. Both beliefs put probability one on $b = 1$, induce terminal threshold 1 and seller continuation $1 - d_S = .99$, and deter every deviation. Prices above $p^\dagger$ have empty weak-gain sets, so D1 imposes no further restriction. The same outward-rounded calculation encloses the endpoint comparisons, overlaps between witness branches, and the four high-price bounds above with strict slack. This completes the criterion-D1 verification in the one-counteroffer contingent-plan representation.

OA4.3 [TENTATIVE] Private Offers during the Bidding Round

A buyer may seek immediate agreement after ordinary bidding has begun, while rivals still have opportunities to raise their bids or join the sale. A private offer can end these opportunities if the seller accepts before another bidder can respond. The relevant comparison is then with continuation from the observed bidding history. We examine this decision at a fixed history, first establishing the conditional gains from agreement and then the buyer's unilateral incentive to offer. Rejection resumes a specified auction in which the buyer remains eligible.

OA4.3.1 Remaining Competition and Gains from Agreement

Let h be the public history of an ascending auction at clock price x , and consider an active buyer whose private type θ includes its match value $b \geq x$. The seller's private information is s . The buyer's remaining benefit from immediate ownership is $d_h(\theta) \geq 0$, a fresh offer costs $\kappa_h > 0$ after either acceptance or rejection, and immediate agreement saves the seller a remaining cost $c_s(h) \geq 0$. Thus an accepted price p gives the buyer $b - p + d_h(\theta) - \kappa_h$. These benefits and costs are measured relative to the remaining auction: preparation costs already incurred cannot be saved, and a benefit available only before bidding began cannot be obtained at h .

After rejection, a committed English-auction continuation preserves the current price floor and the seller's fixed no-sale value v_s . Its effective threshold is $r_s = \max\{v_s, x\}$. Let $Z_1 \geq Z_2$ be the highest and second-highest values among the other buyers who can still affect the outcome, including any exogenous future entrants. Missing rival values are set

to $\underline{b}$, as in the main paper, Section 2, with $x \geq \underline{b}$. Buyers who have dropped out at values below x are absorbed into the price floor. The residual rival distribution is conditional on (h, θ, s) ; it need not retain the original iid form. The stipulated continuation allocates to the highest-value eligible buyer at the larger of r_s and the second-highest buyer value, or leaves the object with the seller if no buyer reaches r_s .

The seller's realized gross payoff P_s^h and the focal buyer's realized surplus U_s^h are therefore

$$\begin{aligned} P_s^h &= \max\{r_s, Z_2, \min\{b, Z_1\}\}, \\ U_s^h &= \max\{b - \max\{r_s, Z_1\}, 0\}. \end{aligned} \tag{OA4.6}$$

If $v_s < x$, the active buyer already clears x , so no sale is impossible; whenever no sale occurs, $r_s = v_s$. Hence the price floor does not assign the seller an artificial no-sale payoff. Write $R_s(h, \theta) = \mathbb{E}[P_s^h \mid h, \theta, s]$ and $u_s(h, \theta) = \mathbb{E}[U_s^h \mid h, \theta, s]$. The seller's net continuation payoff at this fixed type is $R_s(h, \theta) - c_s(h)$.

Proposition OA4.3 (Conditional gains during bidding). *Under the stipulated continuation,*

$$R_s(h, \theta) + u_s(h, \theta) - b = \underbrace{\mathbb{E}[\max\{\max\{r_s, Z_2\} - b, 0\} \mid h, \theta, s]}_{\Gamma_s(h, \theta)} \geq 0. \tag{OA4.7}$$

At a fixed (h, θ, s) with $R_s(h, \theta) \geq c_s(h)$, a nonnegative price accepted with certainty makes both parties weakly better off than this continuation, including the fresh offer cost, if and only if

$$d_h(\theta) + c_s(h) \geq \Gamma_s(h, \theta) + \kappa_h. \tag{OA4.8}$$

Strict inequality permits strict gains for both parties.

Proof. If the focal buyer wins, its payment cancels the seller's receipt and their combined payoff is b . If it loses and either $r_s > b$ or $Z_2 > b$, the seller obtains $\max\{r_s, Z_2\}$ and the buyer obtains zero. In the remaining losing case, one rival exceeds b and pays b . Thus

$$P_s^h + U_s^h - b = \max\{\max\{r_s, Z_2\} - b, 0\}$$

pointwise, including ties. Conditional expectation gives (OA4.7). A mutually acceptable price lies in

$$[R_s(h, \theta) - c_s(h), b + d_h(\theta) - u_s(h, \theta) - \kappa_h].$$

This interval has a nonnegative lower endpoint and is nonempty exactly under (OA4.8); strict inequality gives a nonempty interior. $\square$

The competition wedge therefore applies to the remaining bids as well as to a round that has yet to begin. The proposition evaluates both parties under the same conditional distribution at a fixed buyer type and seller state. With private information, the seller instead assesses the offer using its posterior over the buyer's type; conditional gains alone do not ensure acceptance. Nor do bilateral gains establish a social improvement, since rivals' forgone surplus must also be counted.

OA4.3.2 The Buyer's Local Offer Incentive

Let $\mu_h(s \mid \theta)$ be the buyer's belief over seller information and $a_s(h, p)$ the probability that seller s accepts p . Under the fixed continuation, a rational seller accepts when

$$p \geq \mathbb{E}[R_s(h, \theta) \mid h, s, p] - c_s(h). \quad (\text{OA4.9})$$

The expectation incorporates what the offer reveals about the buyer. Suppose rejection introduces no additional delay, the private offer is neither disclosed nor inferable to rivals, and it changes neither participation nor the residual auction rules. Updating beliefs then leaves truthful private-value bidding and actual continuation payoffs at a fixed type unchanged. Relative to making no offer, the buyer gains

$$\Delta_h(p \mid \theta) = \sum_s \mu_h(s \mid \theta) a_s(h, p) [b + d_h(\theta) - p - u_s(h, \theta)] - \kappa_h. \quad (\text{OA4.10})$$

A price with $\Delta_h(p \mid \theta) > 0$ is a profitable unilateral offer given these seller responses and continuation rules. Acceptance selects seller states, so the forgone continuation payoff must be weighted by acceptance; it cannot generally be replaced by the unconditional mean.

If rejection changes subsequent behavior, denote the buyer's continuation payoff after no offer by $u_s^0(h, \theta)$ and after rejection by $u_s^R(h, \theta, p)$. Direct payoff comparison instead gives

$$\begin{aligned} \Delta_h(p \mid \theta) = & -\kappa_h + \sum_s \mu_h(s \mid \theta) a_s(h, p) [b + d_h(\theta) - p - u_s^0(h, \theta)] \\ & + \sum_s \mu_h(s \mid \theta) [1 - a_s(h, p)] [u_s^R(h, \theta, p) - u_s^0(h, \theta)]. \end{aligned} \quad (\text{OA4.11})$$

The last term matters if the seller uses the rejected offer to change subsequent terms or if further private offers remain possible. The seller's rejection value must likewise include any permitted counteroffers, acceptance of outstanding bids, or further solicitation. Equation (OA4.9) uses the specified auction continuation alone.

OA4.3.3 A Profitable Offer with Three Active Bidders

Consider a history at clock $x = 40$ with three active buyers and no further entrants. Their values were independently uniform on $[0, 100]$, and the observed history reveals only survival to 40. The focal buyer privately has $b = 80$; the two rival values remain independent uniforms on $[40, 100]$. The seller's no-sale value is below 40. The focal buyer has a remaining resolution benefit $d_h = 10$, independently of rival values; a fresh offer costs $\kappa_h = 0.5$, and the seller saves $c = 1$ by accepting. Rejection resumes truthful bidding with no later private-offer opportunity. The seller updates its belief about the focal value after an offer while retaining the stipulated independent rival distribution.

The buyer's continuation surplus and the competition wedge are

$$\begin{aligned} u_h(80) &= \int_{40}^{80} \left(\frac{t-40}{60} \right)^2 dt = \frac{160}{27}, \\ \Gamma_h(80) &= \int_{80}^{100} \left(\frac{100-t}{60} \right)^2 dt = \frac{20}{27}. \end{aligned} \tag{OA4.12}$$

Thus $R_h(80) = 2020/27$. More strongly, the residual payment is pointwise nondecreasing in the focal value, and even at value 100 the seller's expected revenue is only the expected highest rival value, $R_h(100) = 80$. Every posterior over the focal value in $[40, 100]$ that preserves the stipulated rival law therefore gives a net rejection value at most 79. The seller strictly accepts a private offer of $p = 80$.

The buyer obtains $80 + 10 - 80 - 0.5 = 9.5$ from this offer, exceeding its continuation surplus $160/27$ by $193/54 \simeq 3.574$. This is a profitable offer with rational seller acceptance at the specified history despite the seller's uncertainty about the buyer's value. It requires neither an unknown number of buyers nor deterrence of entry: all three buyers are already participating, but the seller does not know how high each would bid if the auction continued.

More generally, the observed number of bidders need not reveal the number still willing to compete or the number yet to arrive. Those uncertainties enter the conditional rival distribution. Progress through the round can both remove rivals and select stronger survivors. In the example, holding $b = 80$ fixed, survival of both rivals to clock 60 raises the wedge from $20/27$ to $5/3$. Remaining resolution gains may change as well, so these comparisons imply no general monotonicity of pre-emption over time. The results condition on the preceding history and a specified opportunity to offer; equilibrium histories and timing would additionally require accounting for all buyers' anticipation of private offers and their competing opportunities to initiate them.

OA5 Additional Examples and Computations

The main paper's welfare examples and their certificate appear in that paper. This section gives an additional aligned-state calibration and the checks for its local robustness result.

OA5.1 Aligned-Composite Three-Action Example

We now illustrate Proposition 3.2 with action support $\{W, P, K\}$. Let all match values be uniform on $[0, 1]$, and take the aligned seller states

$$(n_L, v_L, d_{S,L}) = (2, 0.38, 0.02), \quad (n_H, v_H, d_{S,H}) = (3, 0.42, 0).$$

For $n \geq 0$ late buyers and no-sale continuation value $v > 0$, direct integration under the missing-order-statistic convention gives

$$w_{n,v}(b) = b - \frac{\max\{b^{n+1} - v^{n+1}, 0\}}{n+1}, \quad (\text{OA5.1})$$

$$R_{n,v}(b) = \begin{cases} \frac{n-1}{n+1} + v^n - \frac{n-1}{n+1}v^{n+1}, & b \leq v, \\ \frac{n-1}{n+1} + b^n - \frac{n}{n+1}b^{n+1} + \frac{v^{n+1}}{n+1}, & b > v. \end{cases} \quad (\text{OA5.2})$$

At $n = 0$, these expressions reduce to $w_{0,v}(b) = \min\{b, v\}$ and $R_{0,v}(b) = v$. State H has stronger late participation, a higher no-sale fallback, and a more patient seller, so all three microfoundations raise its net continuation value. The pre-selection no-sale probabilities are $0.38^3 = 0.054872$ and $0.42^4 = 0.031116960$.

Let urgency follow the truncated exponential distribution

$$G(d) = \frac{1 - \exp(-\lambda d)}{1 - \exp(-\lambda \bar{d})}, \quad 0 \leq d \leq \bar{d}, \quad (\text{OA5.3})$$

and take

$$\pi = 0.5, \quad \bar{d} = 1, \quad \lambda = 10, \quad \kappa = 0.05. \quad (\text{OA5.4})$$

The decreasing urgency density represents a market in which most buyers can wait, while a minority faces an acute acquisition or financing deadline.

An outward-rounded interval certificate verifies a fixed point at

$$q_L = 0.609022919, \quad q_H = 0.715349727. \quad (\text{OA5.5})$$

It also verifies $0 < D_L(b) < D_H(b) < \bar{d}$ and $Z(b) > 0$ for every buyer value, so waiting, probing, and knockout all occur with positive probability.

The action masses are

$$(M_W, M_P, M_K) = (0.763873955, 0.100548544, 0.135577501).$$

They imply the following observed outcomes.

Table OA5.1: Pre-emption incidence in the aligned-composite example

Observed outcome	Probability
No early offer	0.763873955
Attempt and rejection	0.050274272
Attempt and completed pre-emption	0.185851773

Consistent with the decreasing urgency density, most buyers wait. Among the minority who act, the higher knockout price accounts for most completed early sales, while probing produces both rejected and accepted attempts. In this calibration, 23.6 percent of buyers attempt pre-emption, and 78.7 percent of attempts succeed. The selected price distribution produces the same sign as the conditional price difference documented by [Flått et al. \(2022\)](#): completed pre-emptions have a higher mean price than completed sales through the planned round.²

The same certificate verifies the strict cross-state response margins—seller H rejects q_L and seller L accepts q_H —the D1 inequalities, and the inward-pointing price-map conditions; the displayed root is locally unique. After accounting for early sales, the state-conditional no-sale probabilities are 0.054348677 and 0.030986927, with prior-weighted probability 0.042667802. Section OA5.2 gives the finite corner reduction, and Section OA5.3 reports the certified rectangles and numerical margins.

The next two subsections give the finite corner argument and the certificate for this example.

OA5.2 Finite Corner Test for the Aligned Example

For the truncated-exponential urgency distribution used below, the four face conditions have a finite corner test. On the interior branch, let $\delta = q_H - q_L$ and $A = 1 - \pi$. Up to a factor common across b ,

$$m_P(b; q) \propto e^{\lambda w_L(b)} - \exp\left[-\lambda\left(\frac{\delta}{\pi} - \frac{\kappa}{A}\right)\right] e^{\lambda w_H(b)}.$$

²In normalized units, weighting q_L and q_H by their state-contingent completion masses gives a mean price of 0.686588 among completed pre-emptions. Integrating $P_s(b)$ over waiting buyers and rejected probes, conditional on sale, gives 0.574124 among completed bidding-round sales. These levels are illustrative. The sign is parameter-dependent: more accepted probing at the lower price reduces the pre-emption mean, while more rejected probing in the high-continuation state raises the bidding-round mean; conditioning on completed sales also changes the composition by removing no-sale histories.

The log-weight score with respect to δ is nondecreasing in b because $w_H(b) - w_L(b)$ is nondecreasing. Since $C_L(b)$ is also nondecreasing, with both functions strict above v_L , the covariance formula for a posterior mean implies that T_P increases in $q_H - q_L$. Similarly, m_K is proportional to $e^{-\lambda[r-w_H(b)]} - e^{-\lambda\bar{d}}$, where $r = [q_H - Aq_L]/\pi$, and its log-weight score implies that T_K increases in r . Hence (B.4) is implied by

$$T_P(\ell_L, \ell_H) > \ell_L, \quad T_P(u_L, u_H) < u_L, \quad T_K(u_L, \ell_H) > \ell_H, \quad T_K(\ell_L, u_H) < u_H.$$

OA5.3 Certificate for the Aligned Uniform Calibration

For the calibration in Section OA5.1, the certified price rectangle is

$$\mathcal{Q}_0 = [.605022919, .613022919] \times [.711349727, .719349727].$$

The truncated-exponential likelihood-ratio comparisons reduce all four continuum face inequalities to the corner evaluations in Section OA5.2. Table OA5.3 reports their strictly positive lower bounds and the global-interiority margins, including support endpoints and both standing-value kinks. Table OA5.2 reports the root box, Krawczyk image, and interval Jacobian. The residual system used for the local calculation is

$$\mathcal{E}(q) = (1 - e^{-10}) \begin{pmatrix} M_P(q)[T_P(q) - q_L] \\ M_K(q)[T_K(q) - q_H] \end{pmatrix}, \quad J = D_q \mathcal{E}.$$

Since both action masses are positive, its zeros coincide with the boundary-price fixed points. Strict Krawczyk inclusion establishes a locally unique zero, and the determinant enclosure excludes zero. At a root, nonsingularity also holds for the Jacobian of $T(q) - q$, since the two residual systems differ by a nonsingular diagonal factor. This certifies the required price box at the displayed primitives, while the corollary's distributional neighborhood remains qualitative: no explicit perturbation radius is computed.

Table OA5.2: Root Enclosure and Local Uniqueness Certificate

Quantity	Outward enclosure
Root-box coordinate q_L	[0.609022919151066, 0.609022919153068]
Root-box coordinate q_H	[0.715349726667131, 0.715349726669133]
Krawczyk image, coordinate L	[0.609022919152066548242, 0.609022919152066548244]
Krawczyk image, coordinate H	[0.715349726668132175635, 0.715349726668132175636]
Jacobian entry J_{11}	[−0.154272567267, −0.154272567240]
Jacobian entry J_{12}	[0.053728588475, 0.053728588488]
Jacobian entry J_{21}	[−0.000041801644, −0.000041801637]
Jacobian entry J_{22}	[−0.135487742935, −0.135487742915]
$\det J$	[0.020904287872, 0.020904287879]

Notes: The calculation uses Arb through `python-flint 0.9.0`, at 60 decimal digits. These intervals are outward-rounded displays of the computed balls. Strict Krawczyk inclusion is checked before display rounding. The Jacobian is for the unnormalised zero-rent equations defined in the text, not for the price map T itself.

Table OA5.3: Uniform Price-Box and Equilibrium Margins

Quantity	Certified lower bound or range
$T_P(\ell_L, \ell_H) - \ell_L$	0.004000000
$u_L - T_P(u_L, u_H)$	0.003999999
$T_K(u_L, \ell_H) - \ell_H$	0.003996513
$u_H - T_K(\ell_L, u_H)$	0.003996068
$\min_{b \in [0,1], q \in \mathcal{Q}_0} D_L(b; q)$	0.020065585
$\min_{b \in [0,1], q \in \mathcal{Q}_0} [\bar{d} - D_H(b; q)]$	0.166323464
$\min_{b \in [0,1], q \in \mathcal{Q}_0} Z(b; q)$	0.011915854
$\ell_L - C_L(0)$	0.165580252
$C_L(1) - u_L$	0.051934414
$\ell_H - C_H(0)$	0.152820206
$C_H(1) - u_H$	0.038429512
$\min_b [C_H(b) - C_L(b)]$	0.068872090
$\mathbb{E}[C_H(b) \mid P] - q_L$	0.085924816
$q_H - \mathbb{E}[C_L(b) \mid K]$	0.086482870
$C_L(1) - q_L$	0.055934414
$C_H(1) - q_H$	0.042429513
Root-box range of D_L	[0.024065585, 0.709022920]
Root-box range of $\tilde{D}_H$	[0.043981439, 0.765349727]
Root-box range of D_H	[0.063897294, 0.821676535]

Notes: The first eleven bounds hold on the entire price rectangle $\mathcal{Q}_0$. The state-ordering bound holds for all buyer values. Seller-response, D1 and cutoff bounds hold on the certified root box. The finite corner argument in Section OA5.2 extends the four face checks to whole faces; the cutoff identities reduce interiority to endpoint checks. All displayed lower bounds are rounded down.

OA5.4 Late-buyer preparation

For this extension of the ban comparison, write O for an offer attempt and A for acceptance, corresponding to I and A_c in the main paper.

Corollary OA5.1 (Late-buyer preparation). *Let $S_s(\tau) \geq 0$ denote the object-specific due-diligence, financing, travel, and related preparation resources that remain unspent when an early offer is accepted at time τ . Holding participation and the terminal auction fixed,*

$$W^B - W^D = \mathbb{E}[\kappa \mathbf{1}\{O\} + \mathbf{1}\{A\}(L_s(b) - d_B - d_{S,s} - S_s(\tau))]. \quad (\text{OA5.6})$$

Preparation incurred before τ , or useful independently of this object, cancels from the comparison. Only object-specific resources that the early sale actually avoids make discretion more attractive.

Proof. Relative to Proposition 4.1, an accepted early sale saves exactly $S_s(\tau)$ in addition to the buyer's resolution benefit and the seller's saved timing cost. Costs already incurred are common to the two policy regimes and cancel. $\square$

Corollary OA5.1 formalizes the timing margin emphasized in Finansdepartementet (2021, ch. 6.6.3) at fixed participation.

OA5.5 A probing calibration

For scale, take values $F = U[0, 1]$, deterministic late participation $N = n_s$, and the aligned state $(n_L, v_L, d_{S,L}) = (2, 0.38, 0.02)$, $(n_H, v_H, d_{S,H}) = (3, 0.42, 0)$, with $\pi = 1/2$ and $\kappa = 0.02$. The exact waiting-only boundary is $\bar{D}_L = 0.02$. The highest-value buyer's probing-knockout cutoff is $\bar{D}_L + \Delta_D/\pi = 0.092821907$, giving the sufficient $\{W, P\}$ interval $0.02 < \bar{d} \leq 0.092821907$. At $\bar{d} = 0.045$, let urgency have the truncated exponential distribution $G(d) = (1 - e^{-10d})/(1 - e^{-10\bar{d}})$ on $[0, \bar{d}]$. The probing price is $q_L = 0.659155058$ and attempts occur with probability 0.073608826. No early offer, a rejected attempt, and successful pre-emption then have probabilities 0.926391174, 0.036804413, and 0.036804413.

OA5.6 Numerical Calibrations

Table OA5.4 summarizes the numerical calibrations. In each calibration based on the baseline model, the designated early buyer and late buyers share the listed value distribution. Write $\text{TE}(\lambda; \bar{d})$ for the truncated-exponential urgency distribution in (OA5.3).

Table OA5.4: Numerical Calibrations

Exercise	Value distribution	Buyer heterogeneity	Other primitives and role
Waiting-probing regime	$F = U[0, 1]$	$G = \text{TE}(10; .045)$	$(n_L, v_L, d_{S,L}) = (2, .38, .02)$, $(n_H, v_H, d_{S,H}) = (3, .42, 0)$, $\kappa = .02$; no offer, rejection, and completed pre-emption all occur.
Aligned full menu	$F = U[0, 1]$	$G = \text{TE}(10; 1)$	The same three-channel aligned states, $\kappa = .05$; probing and knockout offers both occur.
Seller urgency and D1 survival	$F = U[0, 1]$	$G = \text{TE}(10; 1)$	Common $(n, v) = (2, .4)$, $(d_{S,L}, d_{S,H}) = (.02, 0)$, $\kappa = .01$; seller urgency screens offers, and a $\{W, K\}$ equilibrium survives D1.
Ban comparison	F^S and F^T in the welfare comparison in the main paper	$G = 10^{-4}U[0, 1] + (1 - 10^{-4})\text{Beta}(1, 30)$	$(n_L, v_L) = (2, .2)$, $(n_H, v_H) = (6, .4)$, $d_{S,L} = d_{S,H} = 0$, $\kappa = .02$; a ban lowers surplus under F^S and raises it under F^T .
Participation disclosure	$F = U[0, 1]$	$G = \text{TE}(10; 1.2)$	$(n_L, n_H) = (1, 3)$, common $v = .4$, $d_{S,L} = d_{S,H} = 0$, $\kappa = .02$, with $\pi \in \{.65, .775\}$; disclosure raises surplus at $\pi = .65$ and lowers it at $\pi = .775$.
Post-bidding counteroffers	$F = U[0, 1]$	$G = \text{TE}(10; .15)$	$(n_L, n_H) = (2, 3)$, common $v = .4$, $d_{S,L} = d_{S,H} = .01$, $\kappa = .01$; probing survives action-linked counteroffers.
Continued bargaining	$F = U[0, 1]$	Atom-continuous G_η	Positive proposal costs; all three outcomes survive continued bargaining. The type distribution is described in Section OA4.1.
Buyer risk aversion	$F = U[0, 1]$	$d_B = 0$, binary CARA coefficient	$(n_L, n_H) = (2, 7)$, common $v = 0$, $\kappa = .01$; risk aversion supports probing, which survives D1.

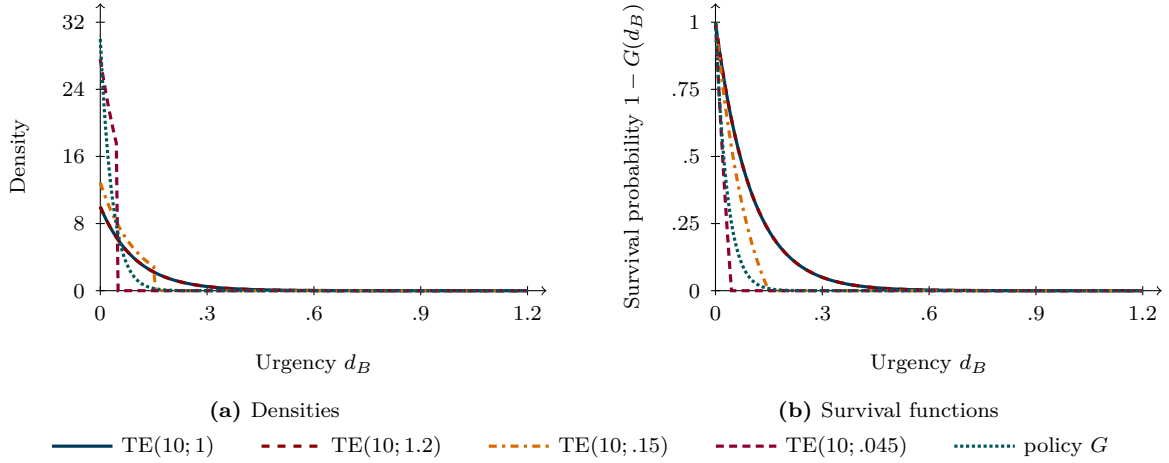


Figure OA5.1: Continuous Urgency Distributions Used in the Numerical Exercises

Notes: $TE(\lambda; \bar{d})$ denotes a rate- λ exponential truncated to $[0, \bar{d}]$. The policy distribution is $10^{-4}U[0, 1] + (1 - 10^{-4})\text{Beta}(1, 30)$. The continued-bargaining specification contains an atom, and the risk-aversion exercise uses two discrete risk-aversion types; neither has a continuous urgency density to plot.

Computational materials. The accompanying `ComputationalSupplement.zip` contains the claim-to-code map, source programs, precision settings, pinned dependencies, and execution records for the numerical certificates and audits. With Python 3.12, install `code/requirements-publication.txt` and run `python code/run_reproduction.py --mode publication` from the package root. The claim map distinguishes interval certificates, exact-arithmetic checks, floating-point audits, and diagnostics. Section OA5.3 displays the aligned-state certificate; its tables and machine-readable interval records are regenerated by `code/generate_aligned_certificate_appendix.py`. The execution records identify software versions, source hashes, commands, and outputs.

OA5.7 Related mechanisms and applications

The mechanism we study is a private binding offer initiated and priced by a buyer before an announced competitive sale. Grebe et al. (2016) study the same Sell-It-Now sequence: one designated buyer makes the offer and, after rejection, enters a second-price auction with one rival. Their risk-neutral benchmark has a publicly known seller value of zero, fixed participation, no reserve or attempt cost, and no direct gains from early resolution. Under their tie-breaking rule, every offer is rejected. Yet 37 percent of Sell-It-Now offers were accepted in their laboratory experiment. Their working-paper version refers to Grebe’s dissertation for an analysis with risk-averse buyers and/or sellers (Grebe et al., 2015, fn. 7). We study how private seller continuation prospects interact with voluntary costly offers and heterogeneous resolution benefits to support no offer, rational rejection, and state-dependent acceptance.

The Sell-It-Now label has also been used more broadly: [Vragov et al. \(2008\)](#) use it for temporary buyer-submitted prices during an ongoing online auction, while developing their formal pricing analysis in seller-posted dynamic-buyout form. The Best Offer label denotes a different sequential haggling game between one seller and a series of potential buyers ([Huang et al., 2013](#)). In [Khezr and Menezes \(2018\)](#), the seller posts a common asking price and buyers simultaneously either accept it or submit counteroffers. In their baseline, the seller then learns its reservation value and may accept the highest counteroffer or retain the object; an auction occurs only when multiple buyers accept the asking price. Acceptance thus depends on seller information, but rejected counteroffers do not lead to an auction. Here the seller knows its continuation state before responding, and rejection preserves the announced auction in which the buyer remains eligible to bid.

Seller-initiated pre-auction offers and buyout prices are studied by [Kirkegaard and Overgaard \(2008\)](#), [Mathews \(2004\)](#), [Ivanova-Stenzel and Kröger \(2008\)](#), and [Khezr \(2018\)](#). [Choné and Linnemer \(2025\)](#) let the seller offer a preferred buyer an option whose rejection informs a later tailored auction; here the informed buyer chooses the price and screens the seller. Indicative bids can privately convey preliminary willingness to pay before costly due diligence, but they are nonbinding and select entrants rather than terminate the sale ([Quint and Hendricks, 2018](#)). Public buyer-initiated pre-emptive bids can deter rival information acquisition or entry, as in [Fishman \(1988\)](#) and [Cramton and Schwartz \(1991\)](#); here rivals do not observe the offer. Private pre-auction messages in [Yan \(2026\)](#) are cheap talk; here the proposed price binds the buyer, and competition ends only if the seller accepts it.

Gains from early trade are already formalized in models with seller-posted prices. [Mathews \(2004\)](#) and [Gallien and Gupta \(2007\)](#) derive buyer equilibria and analyze the seller’s price choice with time-sensitive participants. [Budish and Takeyama \(2001\)](#) and [Mathews and Katzman \(2006\)](#) model insurance gains from buyout options under risk aversion. In these models, the seller sets the early price and buyers choose whether to exercise the option. Reference-dependent preferences can also make a seller-posted price affect subsequent bids ([Shunda, 2009](#)); experimental departures from risk-neutral predictions have been attributed to mistaken beliefs or bounded rationality ([Ivanova-Stenzel and Kröger, 2008](#); [Grebe et al., 2016](#)). In bidder-initiated takeover negotiations, [Calcagno et al. \(2021\)](#) combine bilateral bargaining and learning with latent rival entry; rejection continues bargaining rather than triggering an announced auction, and their published baseline has symmetric information and abstracts from time discounting and negotiation costs. Our baseline combines additive private resolution benefits with a buyer-proposed price that signals the buyer’s counterfactual auction bid and screens the seller’s private continuation state; risk aversion is an extension.

Private target information also appears in takeover models, but at different points in the sequence. In [Baron \(1983\)](#), management responds to the terminal bid after public

tender-offer competition; in [Dimopoulos and Sacchetto \(2014\)](#), an initial public bid may deter rival entry, and the winner learns the target’s minimum acceptable offer only after the auction. A related screening logic outside auctions appears in [Jullien et al. \(2017\)](#): an uninformed buyer offers an early contract to a seller who privately knows future gains from trade. Rejection affects the buyer’s incentive to invest in bypassing the seller, and early contracting can make that threat credible. Here the alternative is an announced auction with competing buyers. The offer also reveals information about the early buyer, who remains a bidder after rejection, so the seller’s assessment of the offer changes its assessment of the auction. This interaction generates rejected attempts and the competition-versus-resolution welfare tradeoff.

Our environment is related to broader comparisons of auctions and negotiations ([Bulow and Klemperer, 1996](#)). In [Bulow and Klemperer \(2009\)](#), sequential buyers can use public pre-emptive bids to influence costly rival entry; our offer is private and late participation is fixed. [McAdams and Schwarz \(2007\)](#) study public buyer offers over repeated rounds when the seller cannot credibly close the sale. Other work on non-commitment, intermediation, and endogenous auction initiation includes [McAfee and Vincent \(1997\)](#), [Akbarpour and Li \(2020\)](#), [Skreta \(2015\)](#), and [Gorbenko and Malenko \(2024\)](#). [Lauermann and Wolinsky \(2026\)](#) study recruitment, entry, and limited commitment in the formation of an informal auction. Their central margin is whether recruitment sustains an auction; ours is whether a privately initiated, binding offer terminates a competitive sale that has already been announced. We hold late participation fixed to isolate buyer selection and the seller’s private continuation value.

Post-auction negotiation can change bidding and extraction ([Koska and Stähler, 2021](#)). Section [OA4.2](#) studies how identifying the winner’s earlier offer decision affects terminal counteroffers and the survival of probing.

Housing-market evidence gives concrete interpretations to several model primitives. Residential transactions can involve extended bilateral bargaining ([Merlo and Ortalo-Magné, 2004](#)), making the time saved by early agreement economically relevant. Bidding wars vary with market conditions ([Han and Strange, 2014](#)), giving the late-participation component of the seller’s continuation state a housing-market interpretation. [Glower et al. \(1998\)](#) find that sellers with a planned move date, and especially an earlier one, sell more quickly; such circumstances correspond to a higher seller-side cost of waiting in our urgency microfoundation. Finally, auction prices primarily reflect buyer values, while negotiated prices place greater weight on slow-moving seller values ([Genesove and Hansen, 2025](#)), supporting the model’s separation between the buyer’s counterfactual bid and the seller’s continuation value.

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