Online Appendix to Pre-empting Competitive Sales

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Overview

The main paper shows why a buyer may make a costly private offer before a competitive sale, why the seller may reject it, and why preventing early agreement can either raise or lower welfare. This appendix examines how those conclusions depend on the source of private information, the reason for valuing early agreement, and the opportunities to continue bargaining.

The nature of seller information matters beyond its effect on the minimum acceptable price. If the seller privately knows that stronger competition is likely, rejection also tells the buyer that the subsequent auction will be harder to win. If the seller instead privately values early resolution, rejection need not change the buyer's auction prospects. These differences determine whether equilibria in which every submitted offer is accepted survive the belief restrictions considered in the paper. We also show why those restrictions can permit several offer prices.

Disclosure addresses a different question from a ban. Revealing verifiable seller information before the buyer acts can avoid offers that would certainly be rejected. But it also changes which early agreements occur, so eliminating failed attempts does not guarantee a welfare improvement. Sellers may prefer to retain their information advantage. These results concern information that can be certified; they do not assume that subjective preferences can generally be verified.

Early agreement can also be valuable because it replaces an uncertain auction payoff with a certain outcome. A risk-aversion extension constructs accepted and rejected offers even without direct benefits from faster resolution. Insurance gains depend on which outcomes an offer avoids and cannot generally be represented by adding the same constant to every buyer's payoff.

Further extensions allow bargaining before the auction or seller counteroffers after bidding. Continued bargaining can attract patient buyers who share in the gains created by buyers who value early agreement. A later seller counteroffer can depend on what the early offer revealed. Additional examples document these mechanisms and explain the calculations used to establish them.

Notation. Notation follows the main paper. The buyer's value is b, its benefit from early resolution is d, and $s \in \{L, H\}$ indexes the seller's continuation state. References to the main paper's numbered statements and equations are marked "main paper" where needed; hyperlinks refer to the companion PDF.

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OA1 Further Equilibrium Regimes and Information Structures

We first examine equilibria in which every submitted offer is accepted, then buyer selection, alternative seller information, and local multiplicity.

OA1.1 Equilibria without Rejection

The support $\{W, K\}$ combines waiting with offers accepted in both states, so no offer is rejected. Unlike the main paper's full-menu equilibrium, it has no probing buyers. Section OA1.4 gives a boundary-price construction; the following result rules out this support under D1 for the aligned continuation state, even without boundary pricing.

Proposition OA1.1 (No refined waiting–knockout equilibrium under the aligned state)**.** *Suppose early and late match values have a common continuous CDF with positive density on $[\underline{b}, \bar{b}]$. Maintain the aligned composite terminal specification: Q_H nontrivially first-order stochastically dominates Q_L and $v_L < v_H$. Suppose $\pi \in (0,1)$, $\kappa > 0$, the resolution benefit is independent of b and has full interval support $[0, \bar{d}]$, the seller state is independent of buyer type, and Assumption 1 holds. Every pure PBE with action support $\{W, K\}$ fails criterion D1, in its strict-gain form, in the corresponding contingent-plan representation.*

The support $\{W, K\}$ can exist in a PBE before refinement, so D1 drives the exclusion. The same conclusion holds under late-participation variation alone. Under fallback variation, however, a top willingness plateau can support a refined $\{W, K\}$ equilibrium: above v_H, the higher no-sale continuation no longer changes buyer willingness. Section OA1.5 proves the proposition and gives this counterexample. Under variation in seller resolution benefits the willingness gap is flat everywhere, and an exact survival condition determines when $\{W, K\}$ survives. For a knockout price q_H, let $b_K^0 = \sup\{b \in [\underline{b}, \bar{b}] : q_H + \kappa - w(b) \geq 0\}$ denote the highest feasible wait–knockout marginal value. Positive waiting and knockout probabilities ensure that this value is defined and that $0 \leq q_H + \kappa - w(b_K^0) < \bar{d}$; it equals $\bar{b}$ whenever $q_H + \kappa \geq w(\bar{b})$.

Proposition OA1.2 (D1 survival with private seller resolution benefits)**.** *Maintain the specification in which only seller resolution benefits vary, with $\kappa > 0$ and $d_{S,L} > d_{S,H} \geq 0$, and fix a pure PBE with support $\{W, K\}$ and price q_H. The equilibrium has a D1-compatible completion in the contingent-plan representation if and only if*

$$d_{S,L} - d_{S,H} \leq C_H(b_K^0) - q_H + \frac{\pi\kappa}{1-\pi} \qquad \text{and} \qquad q_H \leq C_H(b_K^0).$$

When $b_K^0 < \bar{b}$, the first condition reads $d_{S,L} \leq \Gamma(b_K^0) + \kappa/(1-\pi)$, and it implies the second whenever $\Delta_D \geq 0$.

Section OA1.6 proves both directions and reports an interval-certified numerical instance. At the primitives of Example OA1.4, the boundary price $q_H = .637358$ satisfies both survival conditions, with margins .003044 and .013044. D1 therefore permits a refined $\{W, K\}$ equilibrium alongside the $\{W, P, K\}$ equilibrium at these primitives. The survival boundary is $d^*_{S,L} = .023044$; on $(.023044, .060642]$, raw $\{W, K\}$ PBEs exist but fail D1. Among pure assessments with action support $\{W, K\}$, criterion D1 in the contingent-plan representation excludes attempts without rejection under aligned or late-participation information, whereas information about the fallback or seller resolution benefits can preserve this support.

OA1.2 Selection and the Information in Rejection

In supports containing probing, the first attempt cutoff is strictly decreasing in b. At differentiability points,

$$\frac{\partial D_L(b; q_L)}{\partial b} = -w'_L(b) < 0. \tag{OA1.1}$$

Consequently, in $\{W, P\}$ and $\{W, P, K\}$, attempt probability is weakly increasing in b and strictly increasing wherever the cutoff is interior; at differentiability points with an interior cutoff,

$$\frac{\partial \Pr(p > 0 \mid b)}{\partial b} = g(D_L(b))w'_L(b) > 0. \tag{OA1.2}$$

The same monotonicity holds in $\{W, K\}$ because $\partial \widetilde{D}_H(b; q_H)/\partial b = -\bar{w}'(b) < 0$. Under independence of match values and resolution benefits, observing an attempt therefore shifts the seller's posterior toward higher b: a buyer with a better match can justify the fixed offer cost and equilibrium price with a smaller benefit from early resolution.

Within $\{W, P, K\}$, higher-value buyers need a smaller resolution benefit to prefer knockout to probing. At differentiability points,

$$\frac{\partial D_H(b; q)}{\partial b} = -w'_H(b) < 0. \tag{OA1.3}$$

The effect of b on probing itself is generally ambiguous, because a higher value both enters the probing region at its lower boundary and exits it toward knockout at its upper boundary. At differentiability points with interior cutoffs,

$$\frac{\partial \Pr(p = q_L \mid b)}{\partial b} = g(D_L(b))w'_L(b) - g(D_H(b))w'_H(b). \tag{OA1.4}$$

Thus probing becomes more likely with b exactly when the inflow at $D_L(b)$ exceeds the outflow at $D_H(b)$. If resolution benefits are uniform on $[0, \bar{d}]$ and both cutoffs are interior, then

$$\frac{\partial \Pr(p > 0 \mid b)}{\partial b} = \frac{w_L'(b)}{\bar{d}} > 0, \qquad \frac{\partial \Pr(p = q_L \mid b)}{\partial b} = \frac{w_L'(b) - w_H'(b)}{\bar{d}} \leq 0, \qquad \text{(OA1.5)}$$

where the inequality follows from $w_H'(b) \geq w_L'(b)$. More generally, the shape of G determines the strength of selection into offering.

The buyer's terminal surplus is $u_s(b) = \mathbb{E}_s[\max\{b - \max\{v_s, Y_{1N}\}, 0\}]$. Under late-participation variation the state enters through Y_{1N}, so rejection of a probe is bad news about the round the rejected buyer now enters. Under fallback variation the state matters only when the fallback prevents the buyer from clearing the terminal threshold or sets the price under weak late competition; above v_H, the willingness gap is constant in b. Under variation in seller resolution benefits the state never enters u_s: rejection reveals a patient seller but leaves the buyer's auction payoff unchanged.[1]

Selection also changes the interval of prices that separates the seller states. Conditional on probing, this interval has width $\int [C_H(b) - C_L(b)] h_P(b)\, db$. Under variation in seller resolution benefits, the gap is the constant $d_{S,L} - d_{S,H}$, so selection moves both thresholds equally. Under fallback variation, the gap declines with b, so high-value selection narrows it, though it remains bounded below by $\int_{v_L}^{v_H} \phi(F(t))\, dt$. Under late-participation variation, high-value selection can instead preserve or widen the gap: above the common fallback, the derivative of $R_H(b) - R_L(b)$ is $[1 - F(b)][\phi_H'(F(b)) - \phi_L'(F(b))]$, which is positive for sufficiently large $F(b)$.

The ban comparison in Section 4 of the main paper applies to all three sources of seller information. Disclosure requires verifiability; the numerical example below isolates information about participation prospects.

OA1.3 Private Seller Benefits from Early Resolution

When only seller resolution benefits vary, $Q_L = Q_H$ and $v_L = v_H$, so $u_L = u_H$ and $w_L = w_H$. Screening operates through $C_s = R - d_{S,s}$, giving the following explicit probing window.

Corollary OA1.3 (Probing with private seller resolution benefits)**.** *Let Q and v be common across states, with $Q(N \geq 1) > 0$, let $d_{S,L} > d_{S,H} \geq 0$, and write $\Delta z = d_{S,L} - d_{S,H}$ and Γ for the common wedge. The entry thresholds satisfy*

$$\bar{D}_L = \frac{\kappa}{1 - \pi} - d_{S,L}, \qquad \Delta_D = \Delta z - \frac{\pi \kappa}{1 - \pi}.$$

[1] With continued bargaining, seller patience can also affect negotiation before the round.

Suppose

$$\frac{\pi\kappa}{1-\pi} < \Delta z, \qquad d_{S,L} < \min\Big\{R(\underline{b}),\ \Gamma(\underline{b}) + \frac{\kappa}{1-\pi}\Big\}, \qquad \bar{D}_L < \bar{d} \le \frac{(1-\pi)d_{S,L} - d_{S,H}}{\pi}.$$

Then Assumption 1 holds and Proposition 3.1 applies with support $\{W, P\}$: no early offer, a rejected attempt, and successful pre-emption occur with positive probability, the probing offer being accepted exactly by the urgent seller. If in addition $d_{S,L} \le \kappa/(1-\pi)$, the equilibrium has a D1-compatible completion.

Proof. With common (Q, v), $w_H = w_L = w$ and $C_s = R - d_{S,s}$, so $C_H - C_L = \Delta z$ pointwise. Positive late participation gives the strict monotonicity conditions in Assumption 1 by (A.1) and (A.4), while the displayed bound $d_{S,L} < R(\underline{b})$ gives $C_L(\underline{b}) > 0$. Since $\Gamma(\bar{b}) = 0$, $C_s(\bar{b}) = w(\bar{b}) - d_{S,s}$; substituting into (3.5) gives the displayed $\bar{D}_L$ and Δ_D, and $\bar{D}_L + \Delta_D/\pi = [(1-\pi)d_{S,L} - d_{S,H}]/\pi$. The second primitive bound is $\chi_L > 0$; together with $\Delta z > \pi\kappa/(1-\pi)$, it also implies $d_{S,H} < \Gamma(\underline{b}) + \kappa$, or $\chi_H > 0$. The D1 statement is the sufficient condition (C.1) of Corollary OA1.5. □

The gap in seller resolution benefits must exceed the screening premium $\pi\kappa/(1-\pi)$ for a state-splitting price to exist, while seller L's benefit must stay below the level at which even the least attractive buyer could buy it out. The window's upper endpoint is positive exactly when $(1-\pi)d_{S,L} > d_{S,H}$, which the first condition already implies whenever $d_{S,H} \le \kappa$. Beyond the displayed sufficient condition, Corollary OA1.5 gives the exact D1 requirement $\Gamma(b_L^0) \ge d_{S,L} - \kappa/(1-\pi)$ at an interior marginal value b_L^0. The specification in which only seller resolution benefits vary can therefore support both $\{W, P\}$ and $\{W, P, K\}$. The one-price support admits the explicit primitive window above; once knockout is active, the two prices and selected buyer pools must be determined jointly. The following calibration verifies the richer full-menu support.

Example OA1.4 (Private seller resolution benefits as the screening state). Let $F = U[0,1]$, let the common terminal environment have $n = 2$ late bidders and no-sale continuation value $v = .4$, and let the seller benefit from early resolution be $d_{S,s} \in \{0, .02\}$ with equal probabilities, state L being the urgent seller. Let buyer resolution benefits have the rate-10 truncated-exponential distribution on $[0,1]$, and set $\kappa = .01$. An endpoint-inclusive numerical assessment has

$$(q_L, q_H) = (.586757, .643708)$$

and probabilities

$$(.550185, .092644, .357171)$$

of no early offer, a rejected attempt, and successful pre-emption. It satisfies all audited PBE and criterion-D1 conditions in the seller-plan representation.

Private seller resolution benefits can therefore reproduce the baseline screening pattern without private information about late participation. The equilibrium equations are analytical; the example's fixed point and cross-type rankings are numerical. Section OA1.7 gives the proof and audit.

OA1.4 Waiting–knockout construction

For the support defined in Section OA1.1, use the waiting–knockout cutoff and conditional offer probability $m_K(b;q)$ from (B.5). Whenever the aggregate knockout probability is positive, seller H's expected minimum early price is

$$T_K(q) = \frac{\int_{\underline{b}}^{\bar{b}} C_H(b) m_K(b;q) f(b)\, db}{\int_{\underline{b}}^{\bar{b}} m_K(b;q) f(b)\, db}. \tag{OA1.6}$$

The fixed point $q = T_K(q)$ makes seller H indifferent and satisfies the price selection in Assumption 2. Two further checks ensure that both actions occur and that no buyer prefers an unused probe. First,

$$0 < \widetilde{D}_H(\underline{b};q) \qquad \text{and} \qquad \widetilde{D}_H(\bar{b};q) < \bar{d}. \tag{OA1.7}$$

Because the cutoff decreases with b, these inequalities leave fully patient, low-value buyers who wait and sufficiently urgent, high-value buyers who choose knockout. Second, for every buyer type (b,d), require

$$(1-\pi)[w_L(b) + d - C_L(\bar{b})] - \kappa \le \max\{0, \bar{w}(b) + d - q - \kappa\}. \tag{OA1.8}$$

Under belief on $b = \bar{b}$, the cheapest offer accepted by seller L is $C_L(\bar{b})$. Condition (OA1.8) requires its payoff to be no greater than the buyer's best payoff from waiting or knockout.

Local passivity adds a separate nearby-deviation check. An infinitesimal undercut inherits the knockout pool's posterior, so seller H rejects it while seller L accepts it; the undercut is therefore a probe rather than a cheaper knockout. A strict sufficient condition for deterring these local probes is

$$(1-\pi)[w_L(b) + d - q] - \kappa < \max\{0, \bar{w}(b) + d - q - \kappa\} \qquad \text{for every } (b,d). \tag{OA1.9}$$

At a fixed point satisfying (OA1.7)–(OA1.9), Bayes' rule makes seller H indifferent at q, and state ordering makes seller L accept. After sufficiently small undercuts, assign the on-path knockout posterior; the strict inequality (OA1.9) deters the resulting local probes. At all remaining unused prices, assign $b = \bar{b}$. Any unused offer accepted in both states must then cost at least $C_H(\bar{b}) \ge q$, while (OA1.8) deters an offer accepted only in

L. Buyers wait below $\widetilde{D}_H(b;q)$ and offer q above it. These strategies and beliefs form a locally passive PBE with support $\{W,K\}$.

The fixed point exists under simple endpoint conditions. For every $q \in [C_H(\underline{b}), C_H(\bar{b})]$, $T_K(q)$ is a weighted average of $C_H(b)$ and therefore lies in the same interval. Moreover, $\chi_H > 0$ gives $\widetilde{D}_H(\underline{b}; C_H(\underline{b})) > 0$, while $C_H(\bar{b}) - \bar{w}(\bar{b}) + \kappa < \bar{d}$ gives $\widetilde{D}_H(\bar{b}; C_H(\bar{b})) < \bar{d}$. These endpoint inequalities imply positive waiting and knockout probabilities throughout the interval, so T_K is continuous and has a fixed point. A root implements the selected $\{W,K\}$ support when it also satisfies (OA1.8) and (OA1.9). Unlike the $\{W,P\}$ result, these additional checks do not reduce to a single interval for $\bar{d}$: they must deter both distant and infinitesimal probing offers for every buyer type. For example, holding $\bar{d}$ and the waiting–knockout fixed point fixed, a lower continuation value for seller L can make an unused probing offer attractive to intermediate buyer types and invalidate the $\{W,K\}$ support.

The D1 survival conditions are given in Section OA1.1.

OA1.5 Proof of Proposition OA1.1

Let q_H be the knockout price and define the wait–knockout cutoff

$$D_K(b) = q_H - \bar{w}(b) + \kappa, \qquad \mathcal{B}_K = \{b : 0 \le D_K(b) \le \bar{d}\}, \qquad \beta_K = \inf \mathcal{B}_K.$$

Positive waiting and knockout mass, continuity, and full interval support for resolution benefits make this marginal set nonempty, and positive knockout mass gives $\beta_K < \bar{b}$. Write $A = 1 - \pi$, $\Delta(b) = w_H(b) - w_L(b)$, and consider an undercut $p = q_H - \delta$, where $\delta > 0$. By the common-posterior ordering, every rationalizable seller plan $a = (a_L, a_H)$ is nested and belongs to

$$\mathcal{A}(p) \subseteq \{(x,0) : x \in [0,1]\} \cup \{(1,y) : y \in [0,1]\}.$$

Let $\rho(a) = Aa_L + \pi a_H$. At the hypothetical wait–knockout marginal type $(b, D_K(b))$, the deviation gain is

$$m_b(a) = \rho(a)(\delta + \kappa) - A\pi(a_L - a_H)\Delta(b) - \kappa. \tag{OA1.10}$$

For a type with resolution benefit $d = D_K(b) + z$, the equilibrium payoff is $\max\{z, 0\}$, so its deviation gain is exactly

$$h_{b,z}(a) = m_b(a) + \rho(a)z - \max\{z, 0\}. \tag{OA1.11}$$

Under the aligned composite specification, equation (A.3) gives

$$\Delta(b) = 0 \quad \text{for } b \le v_L, \qquad \Delta'(b) = \begin{cases} \phi_L(F(b)) > 0, & v_L < b < v_H, \\ \phi_L(F(b)) - \phi_H(F(b)) > 0, & b > v_H. \end{cases}$$

In the late-participation-only specialization $v_L = v_H = v_0$, the same expressions hold with the middle branch empty, and the argument below applies verbatim. Define

$$\mathcal{B}_K^* = \begin{cases} \{\beta_K\}, & \beta_K \ge v_L, \\ \mathcal{B}_K \cap [\beta_K, v_L], & \beta_K < v_L. \end{cases}$$

For any $b_* \in \mathcal{B}_K^*$, every $b \ge \beta_K$, and every nested plan,

$$m_{b_*}(a) - m_b(a) = A\pi(a_L - a_H)[\Delta(b) - \Delta(b_*)] \ge 0. \tag{OA1.12}$$

The inequality is strict at a partial plan whenever $b > v_L$ and $b_* \le v_L$, or whenever $b > b_* \ge v_L$. Marginal types in $\mathcal{B}_K^*$ have identical gain sets when $\beta_K < v_L$. At full acceptance all marginal types gain the same $\delta > 0$. Moreover, $\rho z - \max\{z, 0\} \le 0$, with strict inequality whenever a nonmarginal type could otherwise tie at a payoff-relevant partial plan. Thus the marginal type at a given b is weakly most willing.

If $b < \beta_K$, then $D_K(b) > \bar{d}$. When $\beta_K > \underline{b}$, continuity gives $D_K(\beta_K) = \bar{d}$, and the deviation payoff is strictly increasing in both b and d whenever $\rho > 0$; rejection alone yields $-\kappa < 0$. Hence the same strict comparison holds below β_K. If $\beta_K = \underline{b}$, that region is empty. Equations (OA1.11)–(OA1.12) therefore leave the single marginal type $(\beta_K, D_K(\beta_K))$ D1-undominated when $\beta_K \ge v_L$. When $\beta_K < v_L$, the D1-undominated set is

$$\Theta_K^* = \{(b, D_K(b)) : b \in \mathcal{B}_K^*\};$$

these plateau types tie rather than delete one another. On the branch $(1, y)$, their common marginal gain is zero at

$$y_K^* = 1 - \frac{\delta}{\pi\{\delta + \kappa + A\Delta(\beta_K)\}} \in (0, 1)$$

for δ sufficiently small, while no type outside the surviving set gains strictly there. Hence no type in the set is itself deleted.

Positive knockout mass and strict increase of $\bar{w}$ give the on-path pool positive probability above $\max\{\beta_K, v_H\}$. Since C_H is constant below v_H and strictly increasing above it, seller H's on-path optimality gives

$$C_H(\beta_K) < \mathbb{E}[C_H(b) \mid q_H] \le q_H.$$

Moreover, $q_H \leq C_H(\bar{b})$: otherwise a slightly lower price above $C_H(\bar{b})$ would be accepted in both states under every posterior and would strictly dominate q_H for its users. Because $C_H(\beta_K) > C_L(\underline{b}) > 0$, choose a positive $p \in (C_H(\beta_K), q_H)$ sufficiently close to q_H. Continuity and strict increase above v_H give a unique $b_p > v_H$ satisfying $C_H(b_p) = p$. A posterior concentrated on b_p makes seller H indifferent and seller L strictly willing to accept, so the whole branch $(1, y)$ is rationalizable. Strict-gain D1 restricts the buyer posterior to the singleton marginal type if $\beta_K \geq v_L$, and to Θ_K^* if $\beta_K < v_L$. In the latter case C_H has the same value $C_H(\beta_K)$ at every permitted type. Every permitted posterior therefore makes both sellers accept p, and every surviving marginal buyer saves $q_H - p > 0$. The $\{W, K\}$ assessment fails D1 in the contingent-plan representation.

Fallback-only boundary. The exclusion is false without late-participation variation. Let $F = U[0, 1]$, let both states have exactly two late buyers, $(v_L, v_H) = (0.2, 0.4)$, $\pi = 1/2$, $d_S = 0.01$, $\kappa = 0.05$, and $d \sim U[0, 0.15]$. The boundary knockout map has a numerically bracketed fixed point $q_H = 0.654466963$, with lowest marginal value $\beta_K = 0.623111832 > v_H$ and knockout mass 0.204305347. For every $b \geq v_H$, $w_H(b) - w_L(b) = 7/375$, so all feasible marginal types have the same D1 gain set. A belief concentrated on the top marginal type makes both sellers reject every undercut because $C_L(1) = 989/1500 > q_H$; no overcut is profitable. Hence this fallback-only knockout equilibrium has positive waiting and knockout mass and admits a D1-compatible completion in the contingent-plan representation.

Parametric diagnostic. Specialize to the late-participation-only case $v_L = v_H = 0$. In the uniform benchmark $b \sim U[0, 1]$, $(n_L, n_H) = (2, 7)$, $\pi = 1/2$, and $d_S = 0.01$, the restrictions

$$\bar{d} > \frac{319}{600}, \qquad \frac{7}{16} < \kappa < \bar{d} - \frac{113}{1200}$$

support a nondegenerate boundary-price PBE with support $\{W, K\}$ before refinement. Indeed, the knockout posterior map continuously maps $[C_H(0), C_H(1)]$ into itself and the upper inequality gives positive knockout mass. The lower inequality also gives

$$\max_b Z(b; C_L(1), q_H) \leq C_H(1) - C_L(1) - \kappa = \frac{5}{24} - \kappa < 0,$$

so the top-value off-path belief deters probing. The proposition then eliminates this raw equilibrium. This appendix calculation is only a diagnostic; the general result does not cover mixed acceptance or an all-knockout assessment.

OA1.6 Proof of Proposition OA1.2

Maintain the specification in which only seller resolution benefits vary: common (Q, v), hence a common w, R, and wedge $\Gamma = R - w$, with $C_s = R - d_{S,s}$ and $d_{S,L} > d_{S,H} \geq 0$.

Fix a pure PBE with support $\{W, K\}$ and price q_H, so a buyer attempts exactly when $w(b) + d \geq q_H + \kappa$ and both seller states accept q_H. Note first that the first survival condition is equivalent to

$$C_L(b_K^0) \geq q_H - \frac{\pi\kappa}{1-\pi}, \tag{OA1.13}$$

since $C_L = C_H - (d_{S,L} - d_{S,H})$ pointwise; when $b_K^0 < \bar{b}$, the marginal identity $w(b_K^0) = q_H + \kappa$ gives $C_s(b_K^0) = q_H + \kappa + \Gamma(b_K^0) - d_{S,s}$, so (OA1.13) reads $d_{S,L} \leq \Gamma(b_K^0) + \kappa/(1-\pi)$, and $\Delta_D \geq 0$ then delivers the second condition because $d_{S,H} \leq d_{S,L} - \pi\kappa/(1-\pi) \leq \Gamma(b_K^0) + \kappa$.

Scalar reduction and the tied curve. Because $w_H \equiv w_L$, a contingent plan $a = (a_L, a_H)$ matters to a deviating buyer only through its acceptance weight $\rho = (1-\pi)a_L + \pi a_H \in [0,1]$: a type with scalar $z = w(b) + d$ obtains $\rho(z-p) - \kappa$ from deviating to p, against the equilibrium value $U^*(z) = \max\{0, z - q_H - \kappa\}$. For $p < q_H$ the weak-gain set is $\{\rho \geq \tau(z;p)\}$ with $\tau(z;p) = [\kappa + U^*(z)]/(z-p)$, strictly decreasing in z below $q_H + \kappa$ and strictly increasing above it. The deviation-eager types after every undercut therefore form the tied curve $\Theta_K = \{(b,d) : w(b) + d = q_H + \kappa\}$, whose value range is $[\beta_K, b_K^0]$ with $b_K^0 = \sup\{b \in [\underline{b}, \bar{b}] : D_K(b) \geq 0\}$. All curve types have identical gain sets, and whenever some rationalizable plan gives the curve types a strict gain, the strict-gain comparison deletes every off-curve type. On the $(x, 0)$ branch this bite occurs exactly when $p < q_H - \pi\kappa/(1-\pi)$, so that $\tau(q_H + \kappa; p) < 1 - \pi$; on the $(1, y)$ branch it extends to every undercut $p \in [C_H(\underline{b}), q_H)$, where a posterior making the patient state indifferent rationalizes an interior y. Whenever this deletion has bite, D1-permitted posteriors are distributions on Θ_K. The top curve type $(b_K^0, D_K(b_K^0))$ remains D1-permitted at every undercut and maximizes $\mathbb{E}[C_s(b)]$ over posteriors on Θ_K.

Survival. Suppose both survival conditions hold. Place the off-path belief on $(b_K^0, D_K(b_K^0))$ at every $p \neq q_H$ and let each seller respond sequentially rationally, accepting at equality. For $p < C_L(b_K^0)$, both states reject and every type's gain is at most $-\kappa < 0$. For $p \in [C_L(b_K^0), q_H)$, the urgent state accepts and the patient state rejects, since $p < q_H \leq C_H(b_K^0)$ by the second condition; thus $\rho = 1 - \pi$, and the supremum over all types of $(1-\pi)(z-p) - \kappa - U^*(z)$ is attained on the curve, where it equals

$$(1-\pi)(q_H + \kappa - p) - \kappa \;\leq\; (1-\pi)\Big(q_H + \kappa - C_L(b_K^0)\Big) - \kappa \;\leq\; 0$$

by (OA1.13). For $p > q_H$, every type's gain is negative under every plan, so gain sets are empty and D1 places no restriction. The belief is D1-permitted at every off-path price and no deviation strictly gains: the equilibrium survives. At equality in (OA1.13) the supremum is zero, weak rather than strict, so the strict-gain criterion still passes; the survival region is closed.

Failure. Suppose (OA1.13) fails. Pick $p \in \Big(C_L(b_K^0),\, q_H - \pi\kappa/(1-\pi)\Big)$, a nonempty interval. Deletion has bite and leaves exactly Θ_K, so under every permitted posterior

$\mathbb{E}[C_L(b)] \le C_L(b_K^0) < p$: the urgent state must accept, giving $\rho \ge 1-\pi$, and every curve type gains at least $(1-\pi)(q_H+\kappa-p)-\kappa > 0$, strictly. If instead the second condition fails, $q_H > C_H(b_K^0)$, any $p \in (C_H(b_K^0), q_H)$ forces acceptance by both states under every permitted posterior, and every knockout type gains $q_H - p > 0$. In either case every D1-consistent completion admits a profitable deviation, so the assessment fails strict-gain D1. □

Numerical instance and boundary. At the primitives of Example OA1.4, the boundary fixed point for support $\{W, K\}$, $q_H = \mathbb{E}[C_H(b) \mid \text{attempt}]$, equals .637358061, with waiting mass .615537 and knockout mass .384463; $b_K^0 = .790989$ and $\Gamma(b_K^0) = .003044$, so both survival conditions hold with margins .003044 and .013044, and this refined equilibrium coexists with the Example's full-menu equilibrium. Because the fixed point does not involve $d_{S,L}$, the survival boundary there is the vertical cut $d_{S,L}^* = \Gamma(b_K^0)+\kappa/(1-\pi) = .023044$: certified failures occur at $d_{S,L} \in \{.025, .03, .05\}$, while raw $\{W, K\}$ PBEs exist up to $d_{S,L} = .060642$, so on $(.023044, .060642]$ the exclusion is driven by criterion D1 exactly as in the aligned case. Raising the offer cost to $\kappa = .05$ yields a variant with $D_K(\bar{b}) \ge 0$ and $C_L(\bar{b}) > q_H = .643707$ (numerically close to, but distinct from, the Example's full-menu price .643708), in which the belief on the top-value marginal type makes both states reject every undercut—the exact analogue of the fallback-only construction above. Because $R' \le w'$ above v, the first-condition residual $q_H - \pi\kappa/(1-\pi) - C_L(b_K^0(q_H))$ is nondecreasing in q_H, so survival of any waiting–knockout price is decided at the lowest, boundary-priced fixed point, and the D1-surviving price set is an interval. An outward-rounded ball-arithmetic calculation applies the Krawczyk theorem to both boundary fixed points—for $\kappa = .01$ in the marginal coordinate b_K^0, and for $\kappa = .05$ directly in the price—and proves existence and local uniqueness with root enclosures of radius below 2×10^{-12}. The same calculation certifies strict signs, with enclosure radii below 10^{-11}, for both survival-condition margins, the survival boundary, the waiting and knockout masses, the partial-acceptance deterrence bound, the killer-interval widths and raw-persistence margins at $d_{S,L} \in \{.025, .03, .05\}$, and the corner and full-rejection margins of the $\kappa = .05$ variant. Every strategic margin used is certified strict except the intended boundary equalities.

OA1.7 Seller Benefits from Early Resolution as the Screening State

Verification of Example OA1.4. Write $A = 1-\pi = \pi = 1/2$. Because the terminal environment is common across seller types, write its buyer willingness and seller revenue as $w(b)$ and $R(b)$. The urgent and patient sellers' minimum early prices are $R(b) - .02$ and $R(b)$. For candidate prices (q_L, q_H), buyer optimality gives cutoffs

$$D_L(b) = q_L - w(b) + \frac{\kappa}{A}, \qquad D_H(b) = \frac{q_H - Aq_L}{\pi} - w(b).$$

The buyer waits below D_L, probes between the cutoffs, and makes the knockout offer above D_H. Setting q_L equal to the urgent seller's posterior minimum early price in the probe pool and q_H equal to the patient seller's posterior minimum early price in the knockout pool gives the displayed fixed point. A 131,072-panel endpoint-inclusive audit gives residuals below 1.7×10^{-14}, action masses

$$(.550185181, .185288486, .264526333),$$

and the positive constant cutoff gap .093902289. At q_L, the urgent seller is indifferent and the patient seller rejects with margin .02. At q_H, the patient seller is indifferent and the urgent seller accepts with the same margin.

For a raw off-path completion, belief on $b = 1$ gives continuation thresholds $(.668, .688)$. Prices below .668 are rejected, prices in $[.668, .688)$ buy only the urgent seller at a higher price than q_L, and prices weakly above .688 buy both types at a higher price than q_H. No deviation is profitable.

For D1, buyer payoffs relative to waiting depend on (b, d) only through $z = w(b) + d$. A contingent seller plan matters only through its aggregate acceptance probability r, and deviation payoff is $r(z - p) - \kappa$. For $.452 \leq p < q_L$, strict-gain sets are ordered by their crossing threshold

$$\frac{\kappa + U^*(z)}{z - p}, \qquad U^*(z) = \max\{0, A(z - q_L) - \kappa, z - q_H - \kappa\},$$

which is minimized at $z_L = q_L + \kappa/A$. D1 leaves all types with this same z_L tied. A belief concentrated on their largest feasible value $b_L^0 = .699522891$ makes both sellers reject because

$$R(b_L^0) - .02 - q_L = .009043008 > 0.$$

For $q_L < p < q_H$, the crossing threshold is minimized at $z_H = (q_H - Aq_L)/\pi$. Its largest feasible value is $b_H^0 = 1$, and

$$R(1) - .02 - q_H = .024292340 > 0.$$

Both sellers again reject. Below .452, rejection is the only rationalizable seller plan, so every buyer's gain set is empty and the first belief remains D1-permitted. Above q_H, the on-path knockout is strictly cheaper than any deviation under the same or narrower acceptance. These regions exhaust the off-path price set and complete the D1 proof. □

OA1.8 Probing refinement beyond the sufficient condition

For a boundary-price waiting–probing equilibrium, let $b_L^0 = \sup\{b : D_L(b; q_L) \geq 0\}$. The sufficient D1 condition in the main paper is part of the following sharper statement.

Corollary OA1.5 (D1 survival)**.** *The waiting-only equilibrium in the first part of Proposition 3.1 admits a D1-compatible belief completion in the contingent-plan representation. For $\kappa > 0$, the on-path profile of a boundary-price waiting–probing PBE in that proposition admits such a completion if and only if $q_L \le C_L(b_L^0)$. The following parameter condition is sufficient:*

$$\bar{D}_L \ge 0. \tag{OA1.14}$$

The undercut argument in the main paper proves sufficiency. If $q_L > C_L(b_L^0)$, a price $p \in (C_L(b_L^0), q_L)$ is accepted by seller L under every surviving posterior and benefits the marginal types. The same strict-gain argument used for the disclosed-state result below establishes necessity. Overcuts are treated as in the main paper.

Boundary Pricing and Probing-Price Multiplicity

Boundary pricing selects a convenient zero-rent member of a larger PBE and D1-compatible set. An exact example uses the late-participation-only uniform specialization

$$(n_L, n_H) = (2, 7), \quad v_L = v_H = 0, \quad \pi = \tfrac{1}{2}, \quad d_S = .01, \quad \kappa = .05, \quad \bar{d} = .15,$$

and the rate-10 truncated-exponential distribution of resolution benefits. Besides the boundary fixed point, consider

$$\widetilde{q}_L = C_L(1) = \frac{197}{300}.$$

The top-value cutoff is $D_L(1; \widetilde{q}_L) = 9/100 < \bar{d}$, while sufficiently low values wait, so both actions have positive mass. The probing pool is nondegenerate below $b = 1$; strict monotonicity of C_L therefore gives

$$\mathbb{E}[C_L(b) \mid \widetilde{q}_L] < C_L(1) = \widetilde{q}_L,$$

and seller L accepts strictly. Seller H rejects because $\widetilde{q}_L < C_H(0) = 37/50$.

Assign every off-path price to $b = 1$. An undercut is rejected because it is below $C_L(1)$, and an accepted L-only overcut is more expensive than the equilibrium probe. The cheapest both-state offer accepted under every seller posterior is $C_H(1)$. For the top type, its payoff relative to probing is

$$V_H(1, \bar{d}; C_H(1)) - V_L(1, \bar{d}; \widetilde{q}_L) = -\frac{29}{1200} < 0,$$

and the difference is weakly smaller for every other buyer type. The complete contingent-plan comparison permits the same endpoint belief: here $b_L^0 = 1$ and the exact D1 condition holds at equality,

$$\widetilde{q}_L = C_L(b_L^0).$$

All remaining seller-response, participation, and deviation inequalities are strict. Continuity therefore yields a nondegenerate interval of waiting–probing, D1-compatible prices immediately below $\widetilde{q}_L$, with seller L earning a positive rent throughout. This establishes local price multiplicity under D1; these prices are outside the locally passive selection in Assumption 2.

OA1.9 Local distributional robustness

Let $\zeta = F^{-1}$ denote the common match-value quantile and T_ζ the associated price map. Call a quantile admissible if it has the maintained support and satisfies the continuity and positive-density assumptions. Holding the support endpoints, participation distributions, no-sale continuation values, and seller benefits from early resolution fixed, the full-menu fixed point in Proposition 3.2 persists under sufficiently small sup-norm perturbations of ζ within this class.

Corollary OA1.6 (Local distributional robustness and the surrounding PBE family)**.** *Let q^0 be a fixed point under an admissible quantile ζ^0 at which the pointwise inequalities in (B.3) are strict. Define $\Psi(\zeta, q) = T_\zeta(q) - q$, and suppose the price Jacobian $D_q\Psi(\zeta^0, q^0)$ is nonsingular. Then, for every admissible quantile ζ sufficiently close to ζ^0 in the sup norm and every sufficiently small rent vector $r = (r_L, r_H) \in \mathbb{R}^2_+$, there is a unique nearby price pair $q(\zeta, r)$ satisfying*

$$q(\zeta, r) - T_\zeta(q(\zeta, r)) = r.$$

It implements a full-menu PBE satisfying (B.3) for every buyer value. In particular, $r = 0$ gives the locally unique nearby boundary-price equilibrium selected by Assumption 2.

The rents are $r_L = q_L - T_{\zeta,P}(q)$ for seller L after probing and $r_H = q_H - T_{\zeta,K}(q)$ for seller H after knockout. Boundary pricing sets both to zero. Small positive rents preserve the action support and acceptance pattern but violate local passivity (Assumption 2). They also change cutoffs, selected buyer pools, and outcome probabilities. Section OA1.10 proves the local result; Section OA5.1 certifies full-menu existence at the aligned example's primitives.

OA1.10 Proof of Corollary OA1.6

Write $b = \zeta(u)$, where u is uniform on $[0, 1]$, and hold (Q_L, Q_H, v_L, v_H) fixed. Let $(U_i)_{i\geq 1}$ be iid uniform draws, independent of $N \sim Q_s$, and let $U_{1N} \geq U_{2N}$ be the two highest

among the first N draws, with $U_{1,0} = U_{2,0} = U_{2,1} = 0$. The auction objects can be written over this fixed probability space as

$$w_s^\zeta(u) = \mathbb{E}[\min\{\zeta(u), \max\{v_s, \zeta(U_{1N})\}\}],$$
$$R_s^\zeta(u) = \mathbb{E}[\max\{v_s, \zeta(U_{2N}), \min\{\zeta(u), \zeta(U_{1N})\}\}].$$

The min and max operators are Lipschitz. Hence w_s^ζ, R_s^ζ, and $C_s^\zeta = R_s^\zeta - d_{S,s}$ vary uniformly continuously with ζ under the sup norm, including when probability moves across either standing value.

In quantile coordinates, the action masses and both coordinates of T_ζ are integrals over a fixed uniform measure. Strict global interiority keeps all cutoffs for resolution benefits in a compact subset of $(0, \bar{d})$ in a neighborhood of (ζ^0, q^0). Since G has a continuously differentiable positive density and both action masses are positive, $\Psi(\zeta, q) = T_\zeta(q) - q$ is continuous in (ζ, q), continuously differentiable in q, and $D_q\Psi$ is jointly continuous. Differentiability with respect to the quantile is not required.

Apply the parameterized implicit-function theorem to

$$\Phi(\zeta, r, q) = T_\zeta(q) - q + r.$$

Nonsingularity of $D_q\Psi(\zeta^0, q^0)$ gives a unique nearby solution $q(\zeta, r)$ to $\Phi = 0$, continuous in (ζ, r). At this solution,

$$q_L - T_{\zeta,P}(q) = r_L, \qquad q_H - T_{\zeta,K}(q) = r_H,$$

so the relevant marginal seller accepts weakly, and strictly when the corresponding rent is positive. Every cutoff, cross-state seller response, positive-mass condition, and endpoint inequality used to deter off-path deviations is strict at $(\zeta^0, 0, q^0)$: positive action masses and strict monotonicity imply $q_L^0 = T_{\zeta^0,P}(q^0) < C_L^{\zeta^0}(\bar{b})$ and $q_H^0 = T_{\zeta^0,K}(q^0) < C_H^{\zeta^0}(\bar{b})$. These inequalities are continuous in (ζ, r, q). The solution therefore remains globally interior and implements the buyer strategy and acceptance pattern in Proposition 3.2. For every admissible common distribution, pointwise $C_L < C_H$ nests seller responses. When $\kappa > 0$, the strict endpoint margins in the nested-plan argument in Appendix C of the main paper persist and supply the stated D1 completion in the contingent-plan representation.

At $r = 0$, the solution is the locally passive boundary-price branch. Positive rent vectors establish the surrounding PBE and D1 robustness result but violate Assumption 2 by the undercut argument following that assumption.

OA2 Disclosure of Verifiable Seller Information

Disclosure reveals the seller's continuation state before the buyer pays to offer. We assume that the relevant information can be certified. If only participation prospects are verifiable, disclosure concerns that component; subjective seller preferences remain private.

OA2.1 Committed Disclosure of the Seller State

Suppose a verifiable signal identifies the seller's continuation state s before the buyer acts and therefore reveals the state-contingent objects $(Q_s, v_s, d_{S,s})$. The result below applies whether the ordering originates in late participation, fallback, seller resolution benefits, or their aligned combination. The numerical application later in the section sets $v_L = v_H$ and $d_{S,L} = d_{S,H}$ to isolate disclosure of participation prospects. Maintain $C_H(b) > C_L(b)$.

A ban removes early contracting; disclosure preserves it while revealing the seller's state. Under automatic disclosure, s becomes public before the buyer acts, through mandatory reporting or a disclosure technology chosen before the seller observes s. Voluntary certification after observing s is considered below.

With a positive attempt cost, a buyer who knows the seller's state never makes an offer certain to be rejected. Disclosure therefore eliminates rejected offers in pure-strategy equilibrium. Its effects on early-sale incidence and welfare remain ambiguous because prices, buyer selection, and seller information rents also change.

Conditional on state s, define

$$\widehat{D}_s(b;q) = q - w_s(b) + \kappa, \tag{OA2.1}$$

$$\widehat{m}_s(b;q) = 1 - G\Big(\widehat{D}_s(b;q)\Big), \tag{OA2.2}$$

and, whenever the denominator is positive,

$$\widehat{T}_s(q) = \frac{\int_{\underline{b}}^{\bar{b}} C_s(b)\widehat{m}_s(b;q)f(b)\,db}{\int_{\underline{b}}^{\bar{b}} \widehat{m}_s(b;q)f(b)\,db}. \tag{OA2.3}$$

The map is the seller's posterior minimum early price among buyers who offer at price q. Let

$$\bar{D}_s^{\mathrm{FD}} = C_s(\bar{b}) - w_s(\bar{b}) + \kappa. \tag{OA2.4}$$

Proposition OA2.1 (Automatic disclosure of the seller state)**.** *Fix a disclosed state s under Assumption 1.*

(i) A no-early-offer PBE exists if and only if

$$\bar{d} \le \bar{D}_s^{\mathrm{FD}}, \tag{OA2.5}$$

and it has a D1-compatible completion.

(ii) If $\bar{d} > \bar{D}_s^{\mathrm{FD}}$, then a no-early-offer PBE does not exist and $\widehat{T}_s$ has a fixed point $\widehat{q}_s \in (C_s(\underline{b}), C_s(\bar{b}))$. There is a PBE in which the buyer waits below $\widehat{D}_s(b; \widehat{q}_s)$, offers $\widehat{q}_s$ above it, and the seller accepts.

(iii) If $\kappa > 0$, every on-path submitted offer is accepted in any pure-strategy PBE under disclosure of the seller state.

(iv) Suppose $\kappa > 0$ and the set of marginal values is nonempty. Define

$$b_s^0 = \sup\left\{b : 0 \leq \widehat{D}_s(b; \widehat{q}_s) \leq \bar{d}\right\}. \tag{OA2.6}$$

The pooling PBE in part (ii) has a D1-compatible completion if and only if

$$\widehat{q}_s \leq C_s(b_s^0). \tag{OA2.7}$$

A sufficient condition is $\widehat{D}_s(\bar{b}; \widehat{q}_s) \geq 0$.

Public disclosure reduces the problem to a state-specific pooling fixed point. With $\kappa > 0$, a knowingly rejected offer is strictly dominated by waiting; Section OA2.2 establishes existence and the D1 boundary.

Combine the two disclosed-state problems by using no-early-offer behavior in a state satisfying (OA2.5) and the pooling equilibrium otherwise. Write FD for this canonical assessment with full disclosure of the seller state, set $\widehat{M}_s = 0$ in a no-early-offer state, and otherwise let

$$\widehat{M}_s = \int_{\underline{b}}^{\bar{b}} \widehat{m}_s(b; \widehat{q}_s) f(b)\, db. \tag{OA2.8}$$

Then

$$\Pr_{\mathrm{FD}}(O) = \Pr_{\mathrm{FD}}(A) = (1-\pi)\widehat{M}_L + \pi \widehat{M}_H, \qquad \Pr_{\mathrm{FD}}(O \setminus A) = 0. \tag{OA2.9}$$

Under opacity, early trade occurs above $D_L(b)$ in state L and above $D_H(b)$ in state H. Disclosure replaces these screening thresholds with the disclosed-state cutoff $\widehat{D}_s(b; \widehat{q}_s) = \widehat{q}_s - w_s(b) + \kappa$. A lower disclosed cutoff brings less urgent buyers—and, at a given benefit from early resolution, buyers with lower match values—into early trade; a higher cutoff instead makes marginal early traders wait. Because the disclosed prices adjust endogenously, either ordering is possible.

The fixed-point condition leaves the seller with zero expected gain relative to the auction within each disclosed-state offer pool. The selected private full-menu equilibrium likewise gives zero rent to the seller state at the margin of each active price. Full disclosure

therefore removes the extra rent seller L earns from a knockout price high enough to induce seller H to accept:

$$U_S^{\mathrm{FD}} - U_S^D = -(1-\pi)M_K \int_{\underline{b}}^{\bar{b}} [C_H(b) - C_L(b)] h_K(b)\, db \leq 0. \tag{OA2.10}$$

Corollary OA2.2 (Ex-ante disclosure commitment)**.** *Suppose that, before observing s, the seller chooses between the selected opaque equilibrium and commitment to the canonical disclosed-state assessment. If disclosure costs $\delta \geq 0$, the seller weakly prefers opacity. The preference is strict if $M_K > 0$ or $\delta > 0$.*

Proof. The disclosure commitment yields $U_S^{\mathrm{FD}} - \delta$. Equation (OA2.10) gives

$$U_S^D - (U_S^{\mathrm{FD}} - \delta) = \delta + (1-\pi)M_K \int [C_H(b) - C_L(b)] h_K(b)\, db \geq 0,$$

with the stated strictness conditions. □

For total surplus, couple the two regimes on the same realization and let A_j and O_j denote acceptance and offer under $j \in \{D, \mathrm{FD}\}$. Since $O_{\mathrm{FD}} = A_{\mathrm{FD}}$,

$$W^{\mathrm{FD}} - W^D = \kappa \Pr_D(O_D \setminus A_D) + \mathbb{E}[(A_{\mathrm{FD}} - A_D)\{d_B + d_{S,s} - L_s(b) - \kappa\}]\,. \tag{OA2.11}$$

The first term is the effort saved by avoiding rejected probes: under opacity the buyer sometimes incurs κ and learns through rejection that the seller has the high continuation state, whereas disclosure reveals that state before the buyer decides whether to make an offer. The second term sums the welfare contributions of the added and removed early sales described by the cutoff changes above. An added early sale raises surplus exactly when $d_B + d_{S,s} > L_s(b) + \kappa$, because its timing gain then exceeds the foregone allocation through competition and the effort cost of the offer; removing an early sale has the opposite sign.

To isolate participation disclosure and the role of the prior, hold $F = U[0,1]$, the common no-sale continuation value $v_0 = .4$, $(n_L, n_H) = (1, 3)$, $d_{S,L} = d_{S,H} = 0$, $\kappa = .02$, $\bar{d} = 1.2$, and $\lambda = 10$ fixed, and change only the probability π of the high-participation state. Table OA2.1 shows that raising π from .65 to .775 reverses the welfare effect of disclosure.

The disclosed state-specific problems do not depend on the prior π, but the opaque screening menu does. At $\pi = .65$, avoiding rejected probes saves .003064, more than the .001446 loss from the changed early-sale set. Raising π to .775 reduces the opaque probing mass and hence the rejection saving to .001947. It also places more weight on the high-participation state, where early agreement displaces the stronger late competition; the total sale-set loss rises to .002283, of which .001607 is the prior-weighted contribution

Table OA2.1: Opposite Welfare Effects of Participation-State Disclosure

Case	$\Pr_D(A)$	$\Pr_{\mathrm{FD}}(A)$	$\Pr_D(O \setminus A)$	Saved rejections	Sale-set effect	$W^{\mathrm{FD}} - W^D$
$\pi = .650$	.22533	.36103	.15322	.003064	-.001446	.001618
$\pi = .775$	.24259	.34158	.09736	.001947	-.002283	-.000336

Notes: "Saved rejections" is the first term in (OA2.11); "Sale-set effect" is the second. Entries are per object. Under disclosure every on-path offer is accepted, so disclosed offer and completion probabilities coincide.

of that state. Section 5 of the Calculation Workbook reports both state contributions. Disclosure then lowers welfare by .000336 even though it eliminates every rejected attempt.

Proposition OA2.1 establishes a state-specific pooling equilibrium, with D1 survival under (OA2.7); accepted price menus may also carry further information about b. Corollary OA2.2 concerns ex-ante commitment to automatic disclosure of the seller state.

OA2.1.1 Interim Voluntary Hard Evidence

Now suppose the seller observes s and then either remains silent or pays δ to present a state-specific certificate. A certificate is feasible only for the matching state. Truthfulness is therefore imposed by the evidence technology, not generated by the seller's reporting incentives. Silence selects the canonical opaque continuation, while a certificate selects the corresponding disclosed-state continuation.

Proposition OA2.3 (Costly voluntary hard evidence)**.** *Conditional on these canonical continuations, for every $\delta > 0$ the unique pure reporting PBE has both seller states remain silent.*

Section OA2.2 proves the result and classifies the zero-cost case, where silence pooling and weak selective disclosure can coexist. The conclusion is conditional on the named continuations, truthful state-specific certificates, and a common seller reporting objective.

Both policy comparisons hold the buyer population, late participation, buyer and seller resolution benefits, and terminal auction fixed. Endogenous entry, preparation, information acquisition, or search would add further equilibrium responses. Continued bargaining also changes rejection values and replaces the single attempt cost with the effort cost of all proposals (Section OA4.1). The welfare accounting must then use the resulting allocation, early agreements, and total costs.

OA2.2 Disclosure Proofs

Proof of Proposition OA2.1. Under no-early-offer behavior, $C_s(\bar{b})$ is the lowest price accepted under every posterior. Because its payoff increases in (b, d), it is unprofitable for all buyers exactly when it is unprofitable at $(\bar{b}, \bar{d})$, yielding (OA2.5). If the inequality

holds, assign lower off-path prices to $b = \bar{b}$; this D1-permitted belief induces rejection. If it fails, types near $(\bar{b}, \bar{d})$ gain from $C_s(\bar{b})$ under every posterior.

If the inequality fails, offer mass is positive at the upper endpoint of $[C_s(\underline{b}), C_s(\bar{b})]$. At $q = C_s(\underline{b})$, buyers in an upper-value set of positive measure submit an offer, and $C_s(b) > C_s(\underline{b})$ on a positive-measure subset because C_s is strictly increasing on a top interval. Hence $\widehat{T}_s(C_s(\underline{b})) > C_s(\underline{b})$. At $q = C_s(\bar{b})$, the posterior mean lies below the upper endpoint, so the intermediate value theorem gives an interior fixed point. Its cutoff is optimal for the buyer, and Bayes' rule makes the seller indifferent at $\widehat{q}_s$. Assign every other price to $b = \bar{b}$: undercuts are rejected, while accepted deviations cost at least $C_s(\bar{b}) > \widehat{q}_s$, so none is profitable.

With public s, a surely rejected offer is dominated by waiting, proving (iii). For (iv), marginal type $(b, \widehat{D}_s(b; \widehat{q}_s))$ needs acceptance probability $\kappa/(\kappa + \widehat{q}_s - p)$ for an undercut $p < \widehat{q}_s$. Types with cutoffs outside $[0, \bar{d}]$ need more and fail the strict-gain test, so the largest surviving minimum early price is $C_s(b_s^0)$. Under (OA2.7), belief on this type rejects every undercut. Otherwise, some $p \in (C_s(b_s^0), \widehat{q}_s)$ is accepted under every surviving posterior and benefits all marginal types. Under the sufficient condition $\widehat{D}_s(\bar{b}; \widehat{q}_s) \geq 0$, positive offer mass also implies $\widehat{D}_s(\bar{b}; \widehat{q}_s) < \bar{d}$. Hence $\bar{b}$ is a marginal value, $b_s^0 = \bar{b}$, and the interior fixed point satisfies $\widehat{q}_s < C_s(\bar{b})$. □

OA2.2.1 Voluntary Hard Evidence

Proof of Proposition OA2.3. Normalize each seller's terminal-auction payoff to zero. Under the selected opaque continuation after pooled silence, seller L earns $K_L^D \geq 0$ and seller H earns zero. Under a correctly matched state-specific continuation, either seller earns zero before paying the certification cost. An H-seller mistaken for L can reject the induced offer and obtain zero; let $J_{LH} \geq 0$ denote an L-seller's payoff from being mistaken for H.

Pooling on silence is a PBE for every $\delta \geq 0$: relative to disclosure, the two seller states gain $K_L^D + \delta$ and δ, respectively. For $\delta > 0$, any profile in which H certifies gives that type $-\delta$, while silence followed by rejection guarantees zero. If only L certifies, it obtains $-\delta$ rather than the nonnegative imitation payoff J_{LH}. Full disclosure of the seller state likewise gives each state $-\delta$, while off-path silence and rejection guarantee zero. Hence no pure reporting profile involving disclosure is a PBE when $\delta > 0$, proving uniqueness within the named continuation class. □

At $\delta = 0$, the inequalities become weak. Besides pooling on silence, H-only disclosure is a PBE. L-only disclosure is supportable if and only if $J_{LH} = 0$, and disclosure by both seller states is supportable when off-path silence is assigned to L and invokes the corresponding state-specific continuation. Reporting-stage D1 and the intuitive criterion do not select between silence pooling and H-only disclosure: each certificate has a singleton

feasible-type set, and H's deviation to silence is an on-path indifference. This appendix classification is conditional on the named offer continuations and does not establish refinement of the reporting profile with disclosure by both seller states.

OA3 Risk Aversion as a Source of Gains from Early Agreement

Risk aversion creates a further gain from early agreement: insurance against an uncertain auction payoff. We first derive the bilateral gain condition, then construct a probing equilibrium with private buyer risk aversion.

OA3.1 Risk Aversion and Equilibrium Selection

Risk aversion changes the comparison between a certain early agreement and the payoff lottery generated by the announced round. Conditional on a known continuation state s, let the buyer's monetary risk premium ρ_B and the seller's monetary risk premium ρ_S equal expected terminal payoff minus its certainty equivalent, on the normalized payoff scale. A larger premium means that the party is willing to surrender more expected payoff to replace the lottery with certainty. A mutually acceptable early price exists exactly when

$$d_B + d_{S,s} + \rho_B + \rho_S \geq \Gamma_s(b) + \kappa. \tag{OA3.1}$$

Resolution benefits and insurance gains must jointly cover $\Gamma_s(b)$ and κ. This condition holds for a known state. In the signaling game, the buyer does not observe s, and probing and knockout insure different lotteries. Their risk premia are therefore action-specific and generally cannot enter the baseline cutoffs as a common additive term.

A positive seller risk premium lowers the seller's minimum acceptable early price, but seller risk aversion also makes its acceptance threshold a nonlinear certainty equivalent of the posterior payoff lottery rather than the posterior mean in (2.7). We therefore establish the general feasibility effect for both parties and construct the signaling equilibrium using privately heterogeneous buyer risk aversion with a risk-neutral seller.

Seller-posted buyout models show that a certain early purchase can insure risk-averse buyers or sellers (Budish and Takeyama, 2001; Reynolds and Wooders, 2009; Mathews and Katzman, 2006). Equation (OA3.1) formalizes the insurance interpretation proposed by Flått et al. (2022); the baseline supplies heterogeneous benefits from early resolution as a second mechanism consistent with a higher conditional pre-emption price. In our buyer-proposed setting, the buyer's privately observed risk aversion determines who offers, while the offered price screens the seller's continuation state. This interaction produces the probing equilibrium below without direct benefits from early resolution.

Proposition OA3.1 (Probing with buyer risk aversion)**.** *Consider the late-participation-only specification with*

$$(n_L, n_H, \pi, \kappa) = (2, 7, .5, .01), \qquad b \sim U[0, 1],$$
$$v_L = v_H = 0, \qquad d_B = d_{S,L} = d_{S,H} = 0.$$

The seller and late bidders are risk neutral. The early buyer has constant absolute risk aversion (CARA) utility $u_\alpha(x) = -e^{-\alpha x}$ *over monetary payoff and privately observes an independent coefficient of absolute risk aversion*

$$\alpha \in \{\alpha_L, \alpha_H\}, \qquad \Pr(\alpha = \alpha_L) = \Pr(\alpha = \alpha_H) = \tfrac{1}{2},$$

where $\alpha_L = .37$ *and* α_H *is the unique root in the certified interval* $[2.78802798023663, 2.78802798043663]$*. There is a waiting–probing PBE at* $q \simeq .655960$*. Every* α_L*-buyer waits, while an* α_H*-buyer offers* q *if and only if* $b \geq .811700$*. Seller L accepts and seller H rejects. The probabilities of no early offer, a rejected attempt, and successful pre-emption are*

$$(.905850, .047075, .047075).$$

The assessment survives criterion D1 over the complete rationalizable seller-plan set.

The auction exposes the early buyer to uncertainty about both whether it wins and how much it pays. A probing offer replaces that lottery with the certain payoff $b - q - \kappa$ in state L, while preserving the auction after rejection in state H. The construction uses the resulting demand for partial insurance: the less-risk-averse class waits, whereas sufficiently high-value buyers in the more-risk-averse class probe. Within either class, the gain from probing rises with match value.

Criterion D1 attributes an unused lower price to the top-value type in the less-risk-averse class and an unused higher price to the top-value type in the more-risk-averse class. The corresponding seller responses make neither deviation profitable. Outward-rounded interval arithmetic certifies the price, match-value cutoff, risk-aversion coefficient, and the continuum inequalities used in this refinement argument. CARA is tractable here because a certain cash shift factors out of exponential expected utility. Section OA3.2 gives the proof and the bilateral-feasibility condition.

OA3.2 Risk Aversion

OA3.2.1 Bilateral Feasibility with Risk Premia

Condition on a known continuation state s. The early buyer's terminal-auction surplus is

$$X_s(b) := \max\{b - \max\{v_s, Y_{1N}\}, 0\},$$

and the seller's terminal monetary payoff is $P_s(b)$ from (2.2). Their expected payoffs are $\mathbb{E}_s[X_s(b)] = b - w_s(b)$ and $\mathbb{E}_s[P_s(b)] = R_s(b)$. Let CE_B and CE_S denote monetary certainty equivalents at the relevant initial wealth. Monetary payoffs use a local normalization for this extension: subtract d_B from every buyer payoff and $d_{S,s}$ from every seller payoff. Increasing each party's initial wealth by the corresponding constant preserves its utility over outcomes. Suppressing dependence on (s, b), define each monetary risk premium as expected net payoff minus its certainty equivalent:

$$\rho_B = \mathbb{E}_s[X_s(b)] - d_B - \mathrm{CE}_B(X_s(b) - d_B), \tag{OA3.2}$$

$$\rho_S = R_s(b) - d_{S,s} - \mathrm{CE}_S(P_s(b) - d_{S,s}). \tag{OA3.3}$$

Concavity implies $\rho_B, \rho_S \geq 0$. The buyer weakly prefers a certain early trade at price p when

$$b - p - \kappa \geq \mathrm{CE}_B(X_s(b) - d_B) = b - w_s(b) - d_B - \rho_B,$$

so its maximum early price is $p_B^{\max} = w_s(b) + d_B + \rho_B - \kappa$. The seller weakly prefers early trade when $p \geq p_S^{\min}$, where

$$p_S^{\min} = \mathrm{CE}_S(P_s(b) - d_{S,s}) = R_s(b) - d_{S,s} - \rho_S.$$

The interval $[p_S^{\min}, p_B^{\max}]$ is nonempty exactly when (OA3.1) holds. The risk premia, buyer resolution benefit, and seller benefit from early resolution increase what the parties can jointly devote to early agreement; the competition wedge and offer cost are what must be financed.

In the signaling game, offers precede knowledge of s. The certainty equivalent of the resulting lottery generally differs from the probability-weighted state-specific certainty equivalents. Probing removes auction risk only in L, whereas knockout removes it in both states. Adding a single risk premium to the baseline cutoffs would therefore require that premium to be the same across states and acceptance profiles.

Risk aversion changes the value of the auction lottery but not the allocation rule: truthful bidding remains weakly dominant in the committed second-price benchmark for any increasing utility function. Thus the extension changes the buyer's ex-ante choice between waiting and offering, not its optimal terminal bid.

OA3.2.2 Buyer Risk Aversion and Equilibrium Sorting

For Proposition OA3.1, specialize to deterministic late-bidder counts $n \in \{2, 7\}$, zero no-sale value, and iid $U[0, 1]$ late values. Write $Y_{1n} \geq Y_{2n}$ for the two highest values among the n late buyers. For a buyer with CARA utility $u_\alpha(x) = -e^{-\alpha x}$, the coefficient $\alpha > 0$ measures absolute risk aversion. Its terminal-auction surplus against n late buyers

is $X_n(b) = \max\{b - Y_{1n}, 0\}$. Define the exponential moment of that surplus and its value-normalized form by

$$M_n(b,\alpha) = \mathbb{E}[e^{-\alpha X_n(b)}], \qquad Z_n(b,\alpha) = e^{\alpha b} M_n(b,\alpha) = \mathbb{E}[e^{\alpha \min\{b, Y_{1n}\}}].$$

Expected terminal utility is $-M_n(b,\alpha)$, so maximizing expected utility is equivalent to minimizing M_n. The normalization Z_n removes the common cash component $e^{-\alpha b}$ and simplifies the comparison of actions. For uniform late values,

$$M_n(b,\alpha) = 1 - b^n + n \int_0^b y^{n-1} e^{-\alpha(b-y)}\, dy.$$

The risk-neutral seller's expected terminal revenue conditional on the early buyer's value is

$$R_n(b) := \mathbb{E}[\max\{Y_{2n}, \min\{b, Y_{1n}\}\}] = \frac{n-1}{n+1} + b^n - \frac{n}{n+1} b^{n+1}.$$

Thus $R_L = R_2$ and $R_H = R_7$ in this late-participation-only specialization. With equal prior probability on the two states, waiting gives the exponential moment

$$W(b,\alpha) = \frac{1}{2} M_2(b,\alpha) + \frac{1}{2} M_7(b,\alpha).$$

A probe at price q is accepted in state L, rejected in state H, and incurs κ in either state. Its exponential moment is therefore

$$P(b,\alpha;q) = e^{\alpha\kappa} \left[\frac{1}{2} e^{-\alpha(b-q)} + \frac{1}{2} M_7(b,\alpha) \right].$$

The buyer prefers probing exactly when $P \le W$. Because expected utility equals the negative of these moments, the following positive rescaling is the buyer's scaled expected-utility gain from probing:

$$\mathcal{F}(b,\alpha;q) = 2e^{\alpha b}\{W(b,\alpha) - P(b,\alpha;q)\} = Z_2(b,\alpha) - e^{\alpha(q+\kappa)} - (e^{\alpha\kappa} - 1) Z_7(b,\alpha).$$

Thus $\mathcal{F} \ge 0$ means probing. Its derivative with respect to match value is

$$\mathcal{F}_b = \alpha e^{\alpha b} \left[(1 - b^2) - (e^{\alpha\kappa} - 1)(1 - b^7) \right].$$

For both calibrated risk-aversion coefficients, $e^{\alpha\kappa} - 1 < 2/7$. Because $(1-b^2)/(1-b^7) \ge 2/7$, we have $\mathcal{F}_b > 0$ for $0 < b < 1$: within either risk-aversion class, a higher match value makes probing more attractive and there is at most one match-value cutoff.

Set $\alpha_L = .37$, and let β denote the match-value cutoff for the more-risk-averse class: an α_H-buyer probes exactly when $b \ge \beta$. We construct a parameter point and equilibrium by solving for (q, β, α_H). The first equation makes the highest-match-value buyer in the

α_L class indifferent, the second makes seller L indifferent at the selected probing price given the pool $b \geq \beta$, and the third makes the buyer with $(b, \alpha) = (\beta, \alpha_H)$ indifferent:

$$\mathcal{F}(1, \alpha_L; q) = 0, \tag{OA3.4}$$

$$q = \frac{1}{1-\beta} \int_\beta^1 R_2(t)\, dt = \frac{3 + \beta + \beta^2 - \beta^3}{6}, \tag{OA3.5}$$

$$\mathcal{F}(\beta, \alpha_H; q) = 0. \tag{OA3.6}$$

Table OA3.1 encloses the solution and reports bounds on the sorting and seller-response margins. The single-crossing bound gives at most one value cutoff within each risk-aversion class. Section 5 of the Calculation Workbook gives the root sign brackets and the remaining global gain comparisons, with their code and interval bounds.

Table OA3.1: Binary-CARA Equilibrium Certificate

Quantity	Outward enclosure or lower bound
Price q	[0.655960451221, 0.655960451222]
Value cutoff β	[0.811699930342, 0.811699930343]
Risk aversion α_H	[2.788027979563, 2.788027980437]
$2/7 - (e^{\alpha_H \kappa} - 1)$	> 0.257441713
$R_2(1) - q$	> 0.010706215
$\mathbb{E}[R_7(b) \mid \text{probe}] - q$	> 0.195440211

Notes: Arb arithmetic uses 80 decimal digits. The price is solved analytically; strict sign brackets and positive derivatives enclose the other two root coordinates. The global D1 comparisons combine analytical reductions with 512 interval subdivisions over each relevant value and undercut domain. The calculation workbook reports the full root residuals, seller-response and gain-ordering bounds. All displayed bounds are rounded outwards; this is not a global equilibrium-uniqueness claim.

Equation (OA3.4) and monotonicity imply that every α_L-type below $b = 1$ waits. Equations (OA3.5)–(OA3.6) imply that α_H-types probe exactly when $b \geq \beta$. Seller L is indifferent and accepts, whereas seller H's rejection margin is .1954402.

For the D1 refinement, the two seller states have disjoint ranges of possible posterior continuation values:

$$R_2(b) \in [1/3, 2/3], \qquad R_7(b) \in [3/4, 7/8].$$

The separation orders all posterior-rationalizable seller responses to an unused price: low prices are rejected in both states, intermediate prices may be accepted only in state L, and sufficiently high prices may be accepted in both states, with mixing possible only at seller indifference. For every potentially profitable price below q, D1 leaves the top-value type in the less-risk-averse class, $\theta_L^* = (1, \alpha_L)$, undominated. Under that belief both sellers reject because $R_2(1) = 2/3 > q$. For potentially profitable prices above q, the corresponding

D1-undominated type is the top-value type in the more-risk-averse class, $\theta_H^* = (1, \alpha_H)$, and seller H rejects. The remaining price intervals offer no buyer type a strict gain. Thus the beliefs

$$\mu(\cdot \mid p) = \begin{cases} \delta_{\theta_L^*}, & p < q, \\ \delta_{\theta_H^*}, & p > q, \end{cases}$$

together with sequentially optimal seller responses, cover the complete price line. The certificate verifies these D1 gain comparisons for every buyer type and every unused price, including the points at which the seller's optimal response changes, as well as every strict seller-response margin. This proves Proposition OA3.1.

OA4 Alternative Bargaining and Sale Protocols

Continued bargaining and seller counteroffers change the payoff from rejecting an early offer and the information used later. We examine each as an alternative to the main paper's committed English-auction continuation.

OA4.1 Pre-emption-Stage Bargaining

The baseline makes the buyer's early offer final: rejection leads directly to the planned auction. Suppose instead that either party may make another binding proposal, at nonnegative proposal costs c_B and c_S, while either party may always end bargaining and activate the unchanged auction. If continued bargaining is valuable to urgent buyers, patient high-value buyers may imitate them and share the resulting information rent. Renegotiation can therefore change both whether early agreement occurs and which buyer types initiate. Formally, if $\Omega_s(p) \geq 0$ is seller type s's incremental value of retaining this option after rejecting p, its effective minimum early price is

$$\mathcal{R}_s(p) = \mathcal{C}_s(p) + \Omega_s(p), \tag{OA4.1}$$

$$\mathcal{R}_L(p) \leq p < \mathcal{R}_H(p). \tag{OA4.2}$$

The second line is necessary for the initial offer to separate the seller states. Full equilibrium additionally requires jointly sequentially rational proposals, exit decisions, posteriors, and continuation play, as in the bargaining literature with incomplete information (Cramton, 1984; Rubinstein, 1985; Grossman and Perry, 1986; Chatterjee and Samuelson, 1987; Fudenberg et al., 1987; Chatterjee and Samuelson, 1988; Cramton, 1992; Watson, 1998); Ausubel et al. (2002) survey this literature.

Proposition OA4.1 (Early resolution and pooling under continued pre-emption-stage bargaining)**.** *Consider the unrestricted alternating-offer extension above, with an initial attempt cost $\kappa > 0$.*

(i) Under a common prior and risk neutrality, if $d_B = d_{S,s} = 0$ almost surely, the auction remains available after every proposal, and perpetual bargaining cannot raise joint material payoff above the auction, every PBE has zero early-attempt probability.

(ii) In the late-participation-only specialization with common $U[0,1]$ private values, there are strictly positive proposal costs and an atom–continuous distribution of buyer resolution benefits for which an unrestricted-game PBE has no early offer, an attempt followed by the auction, and early agreement, each with positive probability. A positive mass of buyers with $d_B = 0$ initiates. No round cap is imposed, although the selected path ends after at most three proposals.

The proof specifies strategies and beliefs after every finite history. Patient high-value buyers pool with urgent buyers and earn information rents through revised offers. Commitment to proceed directly to auction after rejection removes this rent. Pooling can reduce allocative efficiency, while urgent buyers gain from early resolution, so the welfare effect is ambiguous.

OA4.1.1 Proof of Proposition OA4.1

For part (ii), specialize to $v_L = v_H = 0$, deterministic late counts $(n_L, n_H) = (2, 5)$, $d_S = 0$, and $\pi = \Pr(s = H) = 3/10$. Let b and the i.i.d. sequence $(Y_i)_{i\geq 1}$ be mutually independent and uniform on $[0,1]$; the first n_s late buyers participate in state s. Independently draw buyer resolution benefits from

$$G_\eta = \eta\delta_0 + (1-\eta)U[0,1]. \tag{OA4.3}$$

Set

$$\kappa = \frac{1}{2000}, \qquad c_B = \frac{1}{1000}, \qquad c_S = \frac{1}{500}, \qquad \beta = \frac{9}{10},$$

where β is the patient buyer's initiation cutoff, and use initial offer $q = 1/2$, seller counteroffer $r = 3797/5000$, and buyer revision $p = 573/875$. The construction applies for every

$$0 \leq \eta < \eta^\dagger, \qquad \eta^\dagger := \frac{117204341}{120480341}. \tag{OA4.4}$$

Proof. For part (i), let I be the event that the buyer makes a binding initial offer. Fix the realized private types and bargaining history while integrating over terminal-auction randomness. Let $K_B + K_S \geq \kappa$ be the total proposal costs on an attempted history, and define gains relative to the planned auction by $g_B = U_B - u_s(b)$ and $g_S = U_S - C_s(b)$. If bargaining fails, then

$$g_B + g_S = -(K_B + K_S).$$

If agreement occurs, the price cancels between the parties and

$$g_B + g_S = b - u_s(b) - C_s(b) - (K_B + K_S) = -\Gamma_s(b) - (K_B + K_S).$$

The assumed completion gives the same upper bound on an infinite history. Hence every attempted history satisfies $g_B + g_S \leq -\kappa$.

At the buyer's initial information set, waiting gives its conditional expected auction payoff. Initial optimality therefore implies $\mathbb{E}[g_B \mathbf{1}_I] \geq 0$. At the seller's first response, condition on its full information, including its type, the observed proposal, and the preceding contact history. The seller can invoke the unchanged auction before incurring any voluntary continuation cost. Under regular conditional beliefs, defined almost everywhere and updated by Bayes' rule on reached histories, seller optimality and iterated expectations imply $\mathbb{E}[g_S \mathbf{1}_I] \geq 0$. Because both constraints use the common prior,

$$0 \leq \mathbb{E}[(g_B + g_S)\mathbf{1}_I] \leq -\kappa \Pr(I),$$

so $\Pr(I) = 0$.

For part (ii), use the mutually independent construction above. Here $1 - \pi = 7/10$, and Equations (OA5.1) and (OA5.2) give $w_s = w_{n_s,0}$ and $C_s = R_{n_s,0}$. Since $w_L(\beta) = 657/1000$, the stated prices satisfy

$$p = w_L(\beta) - \frac{c_B + \kappa}{1 - \pi} = \frac{573}{875}, \qquad r = (1 - \pi)w_L(\beta) - \kappa + \pi = \frac{3797}{5000}, \qquad q = \frac{1}{2} < p < r.$$

After the seller counteroffer r, termination, revision to p, and acceptance of r yield, respectively,

$$0, \qquad P(b, d) = (1 - \pi)[w_L(b) + d - p] - c_B, \qquad E(b, d) = (1 - \pi)w_L(b) + \pi w_H(b) + d - r.$$

Define

$$a(b) = p + \frac{c_B}{1 - \pi} - w_L(b), \qquad x(b) = a(b) + \frac{\kappa}{1 - \pi} = w_L(\beta) - w_L(b),$$

$$y(b) = \frac{r - (1 - \pi)p - c_B - \pi w_H(b)}{\pi} = 1 - w_H(b).$$

The buyer waits initially when $d < \max\{x(b), 0\}$ and offers q otherwise, with initiation prescribed at the null tie $(\beta, 0)$. Every type that reaches r, including types arriving only through a tremble, follows the complete strategy

$$\begin{cases} \text{terminate}, & d < a(b), \\ \text{revise to } p, & a(b) \le d < y(b), \\ \text{accept } r, & d \ge y(b). \end{cases}$$

Indeed, a equates revision and termination, while y equates acceptance and revision. Moreover,

$$\frac{d}{db}[y(b) - x(b)] = b^5 - b^2 < 0, \qquad y(\beta) > x(\beta) = 0.$$

Thus $x(b) < y(b)$ below β, while $a(b) < x(b) < 0 < y(b)$ above it. Every on-path initiator has $d \ge \max\{x(b), 0\} > a(b)$, so termination is off path. A patient type strictly waits below β and strictly initiates and revises above it.

For the continuous component of resolution benefits, waiting, revision, acceptance of r, and initiation have conditional weights

$$\begin{aligned} e_0(b) &= \max\{x(b), 0\}, \quad m_0(b) = y(b) - \max\{x(b), 0\}, \\ h_0(b) &= 1 - y(b), \qquad\qquad z_0(b) = 1 - \max\{x(b), 0\}. \end{aligned}$$

The patient atom waits below β, revises above it, and never accepts r. Hence

$$\begin{aligned} e_\eta(b) &= (1-\eta)e_0(b) + \eta \mathbf{1}\{b < \beta\}, & m_\eta(b) &= (1-\eta)m_0(b) + \eta \mathbf{1}\{b \ge \beta\}, \\ h_\eta(b) &= (1-\eta)h_0(b), & z_\eta(b) &= (1-\eta)z_0(b) + \eta \mathbf{1}\{b \ge \beta\}. \end{aligned}$$

Integrating over $b \sim U[0,1]$ gives

$$\begin{aligned} Z_e &= \frac{9639 + 26361\eta}{40000}, & Z_p &= \frac{237581 - 153581\eta}{840000}, \\ Z_h &= (1-\eta)\frac{10}{21}, & Z_q &= \frac{30361 - 26361\eta}{40000}, \end{aligned}$$

where $Z_e + Z_p + Z_h = 1$ and $Z_p + Z_h = Z_q$. Bayes' rule makes the post-q density z_η / Z_q and the post-p density m_η / Z_p. For $s \in \{L, H\}$, define the corresponding unnormalized auction values

$$N_{sq} = \int_0^1 z_\eta(b) C_s(b)\, db, \qquad N_{sp} = \int_0^1 m_\eta(b) C_s(b)\, db.$$

After q, both seller states counter at r; after p, seller L accepts and seller H invokes the auction. Before subtracting c_S, the counteroffer payoffs are $V_L = pZ_p + rZ_h$ and $V_H = N_{Hp} + rZ_h$. The complete on-path seller inequalities are

$$V_s > (q + c_S)Z_q, \qquad V_s > c_S Z_q + N_{sq} \quad (s = L, H),$$

$$pZ_p > N_{Lp}, \qquad N_{Hp} > pZ_p.$$

Exact integration gives the six unnormalized margins

$L:$ r versus q	$(97493167 - 88505167\eta)/588000000$
$L:$ r versus auction	$(209471659 - 211036159\eta)/1470000000$
$H:$ r versus q	$(734314608737 - 601422352487\eta)/4095000000000$
$H:$ r versus auction	$(117204341 - 120480341\eta)/16380000000$
$L:$ p versus auction	$(266643493 - 271217293\eta)/5292000000$
$H:$ auction versus p	$(387410369909 + 104670423841\eta)/28665000000000.$

The fourth margin is the first to reach zero, at $\eta = \eta^\dagger$ in (OA4.4); every other margin remains strictly positive there. The stated seller actions are therefore strict throughout the interval in the proposition. At equality the same path is a weak PBE if seller H counters at r when indifferent; above it, H strictly prefers the auction after q.

It remains to complete later histories. Write $\bar{C}_{sq} = N_{sq}/Z_q$ and $\bar{C}_{sp} = N_{sp}/Z_p$. For a preceding on-path buyer offer $j \in \{q, p\}$, let

$$\varepsilon = \frac{c_S - c_B}{2} = \frac{1}{2000}, \qquad \tau_j = \bar{C}_{Lj} + \varepsilon, \qquad T_j = \tau_j + c_B.$$

After an unexpected seller price ζ, the buyer believes state L. Writing $v = w_L(b) + d$, a type with $v > T_j$ accepts when $\zeta \le T_j$ and offers τ_j otherwise; a type with $v \le T_j$ accepts when $\zeta \le v$ and terminates otherwise. These rules compare $v - \zeta$, $v - T_j$, and zero and therefore define a best response for every actual buyer type.

Every type in either on-path posterior has $v \ge w_L(\beta) > T_j$. Following a high seller price, all such types pool on τ_j, and the selected assessment retains the preceding posterior. Seller L strictly accepts by ε, while seller H strictly invokes the auction. The unnormalized versions of

$$w_L(\beta) - T_j, \qquad \bar{C}_{Hj} - \tau_j, \qquad C_L(1) - \tau_j$$

are affine in η and positive at both zero and $\eta^\dagger$; at the latter endpoint their smallest values across $j \in \{q, p\}$ are, respectively, 0.005684921, 0.167132113, and 0.016351588. Moreover, a seller price at or below T_j yields at most

$$T_j - c_S = \bar{C}_{Lj} + \varepsilon + c_B - c_S < \bar{C}_{Lj}.$$

Above T_j, seller L receives $\tau_j - c_S < \bar{C}_{Lj}$, while seller H invokes the auction and receives $\bar{C}_{Hj} - c_S < \bar{C}_{Hj}$. No further seller counteroffer therefore beats the posterior auction value.

At every nonspecial buyer price, use the off-path belief $\delta_{(1,1)}$. The seller's acceptance thresholds are then $C_L(1) = 2/3$ and $C_H(1) = 5/6$. Apply the same response rule to every actual buyer type with

$$\tau_* = C_L(1), \qquad T_* = C_L(1) + c_B.$$

The believed type can use this tail with payoff margin $999/1000$. Seller L accepts τ_* while indifferent to the auction, seller H strictly auctions, and, because $c_S > c_B$, every further seller counteroffer is strictly worse. The four binding global buyer-price comparisons are

$$(1-\pi)[C_L(1) - p] - c_B = \frac{109}{15000}, \qquad C_H(1) - r = \frac{1109}{15000},$$

$$(1-\pi)[C_L(1) - p] = \frac{31}{3750}, \qquad C_H(1) + c_B - r = \frac{281}{3750},$$

and are strictly positive. They cover initial and revised offers accepted only in state L or in both seller states; rejected prices are dominated by not making the proposal. Together with the recursive tail, these inequalities exhaust buyer deviations at every continuation history.

For a Bayes-limit check, preserve atoms on the prescribed q- and p-history tails and add vanishing full-support price densities. At a nonspecial buyer offer, choose type-dependent trembles whose likelihood concentrates on $\{(b,d) : b > 1 - \delta_n,\ d > 1 - \delta_n\}$, $\delta_n \downarrow 0$, while every tremble probability vanishes. The seller's posterior then converges to $\delta_{(1,1)}$. At every finite history h followed by an unexpected seller price ζ, choose positive full-support subdensities $\lambda^n_{L,h}(\zeta)$ and $\lambda^n_{H,h}(\zeta)$ with vanishing total mass and relative rates such that the full L/H likelihood ratio, including the likelihood of h, diverges pointwise in ζ. They depend on the state and public history but not on the buyer's hidden type, and therefore leave the seller's conditional posterior over buyer types unchanged. Apply the construction recursively after every longer finite history. The buyer's posterior then converges to state L after every unexpected seller price. This supplies one atom-preserving, full-support regularization delivering the stated posteriors; it does not establish sequential equilibrium or dynamic D1 refinement.

Finally, for $\eta > 0$, patient buyers initiate with mass $\eta/10$. Their failure and agreement masses are $3\eta/100$ and $7\eta/100$. The total outcome probabilities are

$$\Pr(\text{no early offer}) = \frac{9639 + 26361\eta}{40000},$$

$$\Pr(\text{attempt then auction}) = \frac{237581 - 153581\eta}{2800000},$$
$$\Pr(\text{early agreement}) = \frac{1887689 - 1691689\eta}{2800000}.$$

All three are positive throughout (OA4.4). The buyer strategies, seller inequalities, recursive completion, and Bayes' rule thus define the claimed PBE, completing part (ii). □

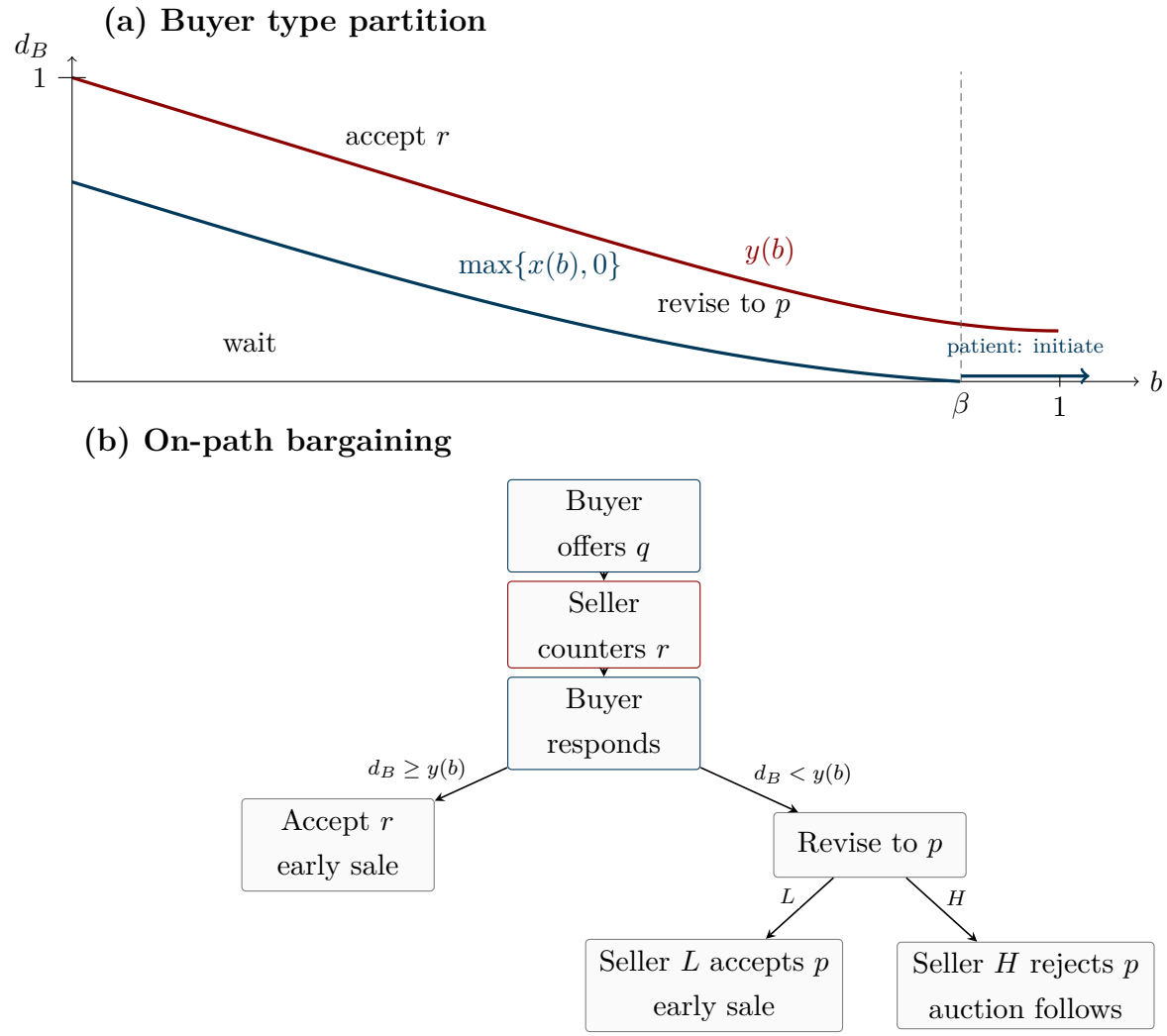


Figure OA4.1: Buyer type partition and on-path proposals in the repeated-bargaining PBE of Proposition OA4.1(ii).

OA4.2 Post-Bidding Counteroffers

A separate commitment concern arises after the bidding round. We study a continuous public English clock followed by one possible seller counteroffer. As a benchmark, the terminal decision cannot condition on the earlier pre-emption action. The seller observes the clock history and the standing price x, set by the last losing dropout, but not the survivor's unsubmitted exit value. It may accept x, or release the survivor from that bid

and make one final counteroffer $r > x$. Rejection of r ends without sale and gives the seller the public fallback v; the survivor cannot revise and losing bidders cannot return (Koska and Stähler, 2021). We assume that the terminal seller has no action-linked selection information and therefore treats the survivor as an otherwise unselected iid draw from F. Conditional on x, the survivor's value then has distribution $F(\cdot \mid \cdot \geq x)$. If

$$r = \rho(v) \in \arg\max_{p \geq v} (p - v)[1 - F(p)]$$

is the unique interior maximizer and $(p - v)[1 - F(p)]$ is weakly decreasing for $p \geq r$, the seller counters at r when $x < r$ and accepts x otherwise, and truthful clock exit remains a weak best response. Thus terminal discretion creates an endogenous, unannounced threshold and can inefficiently exclude willing buyers. The no-action-linkage calculation below derives the survivor's conditional distribution and the resulting continuation objects.

Suppose instead that the seller can identify whether the survivor is the early buyer and, if so, whether that buyer waited or probed. The seller may then condition its counteroffer on that action.

Proposition OA4.2 (Post-bidding counteroffer with action linkage)**.** *Let $F = U[0, 1]$, $(n_L, n_H) = (2, 3)$, $\pi = 1/2$, $v = .4$, $d_{S,L} = d_{S,H} = .01$, $\kappa = .01$, $\lambda = 10$, and $\bar{d} = .15$. There exists a waiting–probing PBE with*

$$(q, r_P, r_W) \simeq (.738406, .780957, .661140),$$

where r_P and r_W are the action-specific counteroffer thresholds after probing and waiting. Its outcome probabilities are

$$\big(\Pr(\textit{no early offer}), \Pr(\textit{rejection}), \Pr(\textit{success})\big) \simeq (.865398, .067301, .067301).$$

Every on-path initiator has a positive benefit from early resolution. The assessment survives criterion D1 in the stated one-counteroffer contingent-plan representation, over the complete set of posterior-rationalizable early responses and sequentially optimal one-counteroffer rules, including irregular posterior-revenue rules.

The analysis below gives the fixed-point system and refinement proof. An interval certificate establishes the root, optimality of both on-path counteroffers, and the D1 inequalities. Thus probing survives even when the seller tailors its terminal counteroffer to the winner's early action.

OA4.2.1 No Action Linkage

When the terminal decision cannot condition on the earlier action, let $X_{(m)} \geq X_{(m-1)}$ denote the two highest values among the $m \geq 2$ active bidders. Conditional on the final English-clock price $X_{(m-1)} = x$, the survivor's value $X_{(m)}$ has density

$$f_{X_{(m)}|X_{(m-1)}=x,m}(y) = \frac{f(y)}{1-F(x)}\mathbf{1}\{y \geq x\}.$$

This density is independent of m. For $x < \bar{b}$, a terminal price $p \geq \max\{x, v\}$ gives expected seller payoff $v + (p-v)[1-F(p)]/[1-F(x)]$. Thus, for a general distribution, the seller chooses

$$\sigma(x) \in \arg\max_{p \geq \max\{x,v\}} (p-v)[1-F(p)].$$

Under the post-peak monotonicity condition stated above, r maximizes this expression when $x < r$, and x is a maximizer when $x \geq r$. Choosing acceptance in the latter case gives $\sigma(x) = \max\{x, r\}$. Uniqueness of the unconstrained maximizer alone does not imply this rule: revenue could rise again beyond a standing price above r. The continuation formulas below apply to the threshold rule; the constrained maximization remains valid without the shape restriction.

Let $Y_{1,n} \geq Y_{2,n}$ be the two highest values among n late buyers, and write $Y_{1s} = Y_{1,n_s}$. Define

$$R_F(n, b; r) := \mathbb{E}[\max\{r, Y_{2,n}, \min\{b, Y_{1,n}\}\}],$$

the seller's gross expected payoff when both the committed threshold and the no-sale payoff equal r. With counteroffer threshold r, the early buyer's willingness and the seller's continuation objects are

$$w_s^{CO}(b) = b - \mathbb{E}[\max\{b - \max\{r, Y_{1s}\}, 0\}],$$

$$R_s^{CO}(b) = R_F(n_s, b; r) - (r-v)F(r)^{n_s}\mathbf{1}\{b < r\}, \qquad C_s^{CO}(b) = R_s^{CO}(b) - d_S. \quad \text{(OA4.5)}$$

OA4.2.2 Action Linkage

Suppose instead that the terminal seller can identify an early-buyer survivor and remember whether that buyer waited or probed. If $\alpha_a(b)$ is the probability of action $a \in \{W, P\}$ at value b, define the unnormalized selected tail

$$T_a(z) = \int_z^1 \alpha_a(t)\, dF(t).$$

After the last losing bidder drops out at x, the action-specific terminal price satisfies

$$\sigma_a(x) \in \arg\max_{z \geq \max\{x,v\}} (z - v)T_a(z).$$

The normalizing probability $T_a(x)$ cancels from this maximization. Thus the early action affects both the terminal counteroffer and the buyer's payoff after a rejected early offer.

At the uniform primitives of Proposition OA4.2, let $Y_{1,n} \geq Y_{2,n}$ be the two highest values among n late buyers and let $\rho = (1 + v)/2$. For a sequential terminal-price rule σ with $\sigma(x) \geq \max\{x, v\}$, define

$$S_n^\sigma(b) = \mathbb{E}[\max\{b - \sigma(Y_{1,n}), 0\}].$$

The seller's corresponding gross terminal payoff conditional on b is $R_n^\sigma(b) = \mathbb{E}[P_n^\sigma(b)]$, where

$$\begin{aligned} P_n^\sigma(b) = \mathbf{1}\{Y_{1,n} \leq b\}\Big[\mathbf{1}\{b \geq \sigma(Y_{1,n})\}\sigma(Y_{1,n}) + \mathbf{1}\{b < \sigma(Y_{1,n})\}v\Big] \\ + \mathbf{1}\{Y_{1,n} > b\}\Big[\mathbf{1}\{Y_{1,n} \geq \rho\}\max\{\rho, b, Y_{2,n}\} + \mathbf{1}\{Y_{1,n} < \rho\}v\Big]. \end{aligned}$$

For the scalar rule $\sigma_r(x) = \max\{x, r\}$, write $S_n(b; r) = S_n^{\sigma_r}(b)$, $R_n(b; r) = R_n^{\sigma_r}(b)$, and

$$S_n(b; r) = \frac{\max\{b^{n+1} - r^{n+1}, 0\}}{n + 1}, \qquad w_s(b; r) = b - S_{n_s}(b; r), \qquad C_s(b; r) = R_{n_s}(b; r) - d_S.$$

Thus r changes the counteroffer faced by an early-buyer winner; a late winner still faces the common threshold ρ.

For the waiting–probing assessment, let r_P and r_W denote the selected terminal thresholds after probing and waiting. In this subsection, subtract d from every payoff of buyer type (b, d) and d_S from every seller payoff. This local normalization leaves all choices unchanged. Let $\tau = (b, d)$ denote a buyer type and set $A = 1 - \pi$. Its on-path waiting and probing payoffs are

$$\begin{aligned} U_W(b, d) &= AS_{n_L}(b; r_W) + \pi S_{n_H}(b; r_W) - d, \\ U_P(b, d) &= A(b - q) + \pi\{S_{n_H}(b; r_P) - d\} - \kappa, \end{aligned} \qquad U^*(b, d) = \max\{U_W(b, d), U_P(b, d)\}.$$

We also write $U^*(\tau) = U^*(b, d)$. The wait–probe cutoff for resolution benefits and probing probability are

$$\begin{aligned} D(b) &= q - w_L(b; r_W) + \frac{\kappa - \pi\{w_H(b; r_W) - w_H(b; r_P)\}}{A}, \\ \alpha_P(b) &= 1 - G(D(b)). \end{aligned}$$

The full-linkage fixed point solves

$$q = \frac{\int \alpha_P(b) C_L(b; r_P)\, dF(b)}{\int \alpha_P(b)\, dF(b)},$$
$$r_P \in \arg\max_{r \geq v} (r - v) \int_r^1 \alpha_P(b)\, dF(b),$$
$$r_W \in \arg\max_{r \geq v} (r - v) \int_r^1 [1 - \alpha_P(b)]\, dF(b).$$

At the primitives in Proposition OA4.2, an outward-rounded ball-arithmetic calculation applies the Krawczyk theorem to the system above and proves a root in the box centered at

$$(q, r_P, r_W) = (.738406414005472, .780957223173591, .661140319939824)$$

with radius 10^{-10} in each coordinate. The Krawczyk image has coordinate radii below 5.8×10^{-17}, so the root is unique within this box. Interval differentiation also establishes that each selected posterior-revenue derivative is strictly decreasing on its support. The two root coordinates are therefore the global maxima of their respective posterior-revenue curves. Hence $\sigma_a(x) = \max\{x, r_a\}$ is sequentially optimal after every on-path clock dropout.

After an unused price p, let ν be the marginal posterior over b, let $T_\nu(z) = \nu([z, 1])$, and let a receiver plan be $\alpha = (\sigma, a_L, a_H)$, where

$$\sigma(x) \in \arg\max_{z \geq \max\{x, v\}} (z - v) T_\nu(z)$$

and a_s is the probability that seller state s accepts the early price. Define

$$K_n(\nu, \sigma) = \int R_n^\sigma(b)\, d\nu(b) - d_S.$$

Sequential rationality requires $a_s = 1$ if $p > K_{n_s}(\nu, \sigma)$, $a_s = 0$ if $p < K_{n_s}(\nu, \sigma)$, and permits mixing at equality. Type $\tau = (b, d)$'s deviation payoff is

$$\begin{aligned} u_\tau(p; \alpha) = A\{a_L(b - p) + (1 - a_L)[S_{n_L}^\sigma(b) - d]\} \\ + \pi\{a_H(b - p) + (1 - a_H)[S_{n_H}^\sigma(b) - d]\} - \kappa. \end{aligned}$$

A scalar-reserve witness uses $\sigma = \sigma_r$.

It remains to verify off-path incentives when the same posterior determines early acceptance and the subsequent terminal rule. Let

$$\tau_M = (1, D(1)), \qquad D(1) \simeq .040637366, \qquad \tau_U = (1, \bar{d}),$$

and let

$$p^\dagger = 1 - \kappa - U^*(1, \bar{d}) \simeq .865699572.$$

Equal-revenue posterior tails and mixing at seller indifference generate scalar-reserve witness plans. In the following table, (a_L, a_H) are early-acceptance probabilities and x is the mixing probability of the indifferent seller state:

unused-price interval	terminal threshold	(a_L, a_H)	undominated type
$p \le .644733060$	.632096261	$(0, 0)$	τ_M
$p \in [.644733060, .727669887]$	r_W	$(x, 0)$	τ_M
$p \in [.727669887, q)$	.678	$(x, 0)$	τ_M
$p \in (q, .824480449]$	r_P	$(1, x)$	τ_U
$p \in [.80, p^\dagger]$	.85	$(1, x)$	τ_U.

In each row $x \in [0, 1]$ makes the displayed type indifferent, while no other buyer type gains strictly. The intervals overlap and cover every unused price at or below $p^\dagger$. Consequently, enlarging the receiver's plan set cannot D1-dominate the displayed type: its witness plan would have to belong to a proposed dominator's strict-gain set, but it belongs to no type's strict-gain set.

For completeness, consider an arbitrary posterior ν as above, possibly with several posterior-revenue peaks. Define

$$\Phi_\nu(z) = (z - v)T_\nu(z), \qquad r_\nu = \min \arg\max_z \Phi_\nu(z), \qquad M_\nu(x) = \max_{z \ge \max\{x, v\}} \Phi_\nu(z).$$

Every sequentially optimal counteroffer is at least $\max\{x, r_\nu\}$, and therefore

$$S_n^\sigma(b) \le S_n(b; r_\nu).$$

Let Λ_n be gross seller revenue when the early buyer's value is zero:

$$\Lambda_n = v\rho^n + n(1-\rho)\rho^n + (n-1)(1-\rho^n) - \frac{n(n-1)}{n+1}(1-\rho^{n+1}).$$

Conditioning on the highest late value and integrating the late-winner term by parts gives

$$\begin{aligned} K_n(\nu, \sigma) + d_S = \Lambda_n &+ \int_0^1 n x^{n-1} M_\nu(x)\, dx \\ &+ \int_\rho^1 n x^{n-1}(1 + v - 2x) T_\nu(x)\, dx. \end{aligned}$$

The final integrand is nonpositive, while $M_\nu(x) \le (r_\nu - v)T_\nu(r_\nu) \le r_\nu - v$. Hence

$$K_n(\nu, \sigma) \le \Lambda_n + r_\nu - v - d_S$$

for every posterior-optimal one-counteroffer rule, without a regularity or scalar-reserve restriction.

Here $(\Lambda_{n_L}, \Lambda_{n_H}) = (.562, .62005)$. At $p = p^\dagger$, rejection by L requires $r_\nu \geq p^\dagger - .152 \simeq .713699572$, while rejection by H requires $r_\nu \geq p^\dagger - .21005 \simeq .655649572$. Combining these floors with the buyer-surplus bound gives the following rigorous upper bounds:

(a_L, a_H)	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$
$\sup_\tau\{u_\tau(p^\dagger;\alpha) - U^*(\tau)\}$	0	$< -.036828520$	$< -.032651821$	$< -.008706110.$

Higher unused prices raise the relevant threshold floor and make accepted trade more expensive, so no type weakly gains above $p^\dagger$.

Finally, assign belief δ_{τ_M} below q and δ_{τ_U} above q through $p^\dagger$. Both beliefs put probability one on $b = 1$, induce terminal threshold 1 and seller continuation $1 - d_S = .99$, and deter every deviation. Prices above $p^\dagger$ have empty weak-gain sets, so D1 imposes no further restriction. The same outward-rounded calculation encloses the endpoint comparisons, overlaps between witness branches, and the four high-price bounds above with strict slack. This completes the criterion-D1 verification in the one-counteroffer contingent-plan representation.

OA5 Additional Examples and Computations

This section adds an aligned-state calibration, its existence certificate, and supplementary policy calculations. The paired ban examples and their certificate remain in the main paper.

OA5.1 Aligned-Composite Three-Action Example

We now illustrate Proposition 3.2 with action support $\{W, P, K\}$. Let all match values be uniform on $[0,1]$, and take the aligned seller states

$$(n_L, v_L, d_{S,L}) = (2, 0.38, 0.02), \qquad (n_H, v_H, d_{S,H}) = (3, 0.42, 0).$$

For $n \geq 0$ late buyers and no-sale continuation value $v > 0$, direct integration under the missing-order-statistic convention gives

$$w_{n,v}(b) = b - \frac{\max\{b^{n+1} - v^{n+1}, 0\}}{n+1}, \tag{OA5.1}$$

$$R_{n,v}(b) = \begin{cases} \dfrac{n-1}{n+1} + v^n - \dfrac{n-1}{n+1}v^{n+1}, & b \leq v, \\ \dfrac{n-1}{n+1} + b^n - \dfrac{n}{n+1}b^{n+1} + \dfrac{v^{n+1}}{n+1}, & b > v. \end{cases} \tag{OA5.2}$$

At $n = 0$, these expressions reduce to $w_{0,v}(b) = \min\{b, v\}$ and $R_{0,v}(b) = v$. State H has stronger late participation, a higher no-sale fallback, and a more patient seller, so all three microfoundations raise its net continuation value. The pre-selection no-sale probabilities are $0.38^3 = 0.054872$ and $0.42^4 = 0.031116960$.

Let the benefit from early resolution follow the truncated exponential distribution

$$G(d) = \frac{1 - \exp(-\lambda d)}{1 - \exp(-\lambda \bar{d})}, \qquad 0 \le d \le \bar{d}, \tag{OA5.3}$$

and take

$$\pi = 0.5, \qquad \bar{d} = 1, \qquad \lambda = 10, \qquad \kappa = 0.05. \tag{OA5.4}$$

The decreasing density of resolution benefits represents a market in which most buyers can wait, while a minority faces an acute acquisition or financing deadline.

An outward-rounded interval certificate verifies a fixed point at

$$q_L = 0.609022919, \qquad q_H = 0.715349727. \tag{OA5.5}$$

It also verifies $0 < D_L(b) < D_H(b) < \bar{d}$ and $Z(b) > 0$ for every buyer value, so waiting, probing, and knockout all occur with positive probability.

The action masses are

$$(M_W, M_P, M_K) = (0.763873955, 0.100548544, 0.135577501).$$

They imply the following observed outcomes.

Table OA5.1: Pre-emption incidence in the aligned-composite example

Observed outcome	Probability
No early offer	0.763873955
Attempt and rejection	0.050274272
Attempt and completed pre-emption	0.185851773

Consistent with the decreasing density of resolution benefits, most buyers wait. Among the minority who act, the higher knockout price accounts for most completed early sales, while probing produces both rejected and accepted attempts. In this calibration, 23.6 percent of buyers attempt pre-emption, and 78.7 percent of attempts succeed, giving completed pre-emption in 18.6 percent of model sales opportunities. This example illustrates equilibrium selection and the conditional price ordering; it is not calibrated to the empirical incidence of kupping. The selected price distribution produces the same

sign as the conditional price difference documented by Flått et al. (2022): completed pre-emptions have a higher mean price than completed sales through the planned round.[2]

The same certificate verifies the strict cross-state response margins—seller H rejects q_L and seller L accepts q_H—the D1 inequalities, and the inward-pointing price-map conditions; the displayed root is locally unique. After accounting for early sales, the state-conditional no-sale probabilities are 0.054348677 and 0.030986927, with prior-weighted probability 0.042667802. Section OA5.2 gives the finite corner reduction, and Section OA5.3 reports the certified rectangles and numerical margins.

OA5.2 Finite Corner Test for the Aligned Example

For the truncated-exponential distribution of resolution benefits used below, the four face conditions have a finite corner test. On the interior branch, let $\delta = q_H - q_L$ and $A = 1 - \pi$. Up to a factor common across b,

$$m_P(b; q) \propto e^{\lambda w_L(b)} - \exp\left[-\lambda\left(\frac{\delta}{\pi} - \frac{\kappa}{A}\right)\right] e^{\lambda w_H(b)}.$$

The log-weight score with respect to δ is nondecreasing in b because $w_H(b) - w_L(b)$ is nondecreasing. Since $C_L(b)$ is also nondecreasing, with both functions strict above v_L, the covariance formula for a posterior mean implies that T_P increases in $q_H - q_L$. Similarly, m_K is proportional to $e^{-\lambda[r - w_H(b)]} - e^{-\lambda \bar{d}}$, where $r = [q_H - A q_L]/\pi$, and its log-weight score implies that T_K increases in r. Hence (B.4) is implied by

$$T_P(\ell_L, \ell_H) > \ell_L, \quad T_P(u_L, u_H) < u_L, \quad T_K(u_L, \ell_H) > \ell_H, \quad T_K(\ell_L, u_H) < u_H.$$

OA5.3 Certificate for the Aligned Uniform Calibration

For the calibration in Section OA5.1, the certified price rectangle is

$$\mathcal{Q}_0 = [.605022919, .613022919] \times [.711349727, .719349727].$$

The truncated-exponential likelihood-ratio comparisons reduce all four continuum face inequalities to the corner evaluations in Section OA5.2. Table OA5.3 reports their strictly positive lower bounds and the global-interiority margins, including support endpoints

[2] In normalized units, weighting q_L and q_H by their state-contingent completion masses gives a mean price of 0.686588 among completed pre-emptions. Integrating $P_s(b)$ over waiting buyers and rejected probes, conditional on sale, gives 0.574124 among completed bidding-round sales. Section 4 of the Calculation Workbook reports interval bounds for both means and their completed-sale denominators. These levels are illustrative. The sign is parameter-dependent: more accepted probing at the lower price reduces the pre-emption mean, while more rejected probing in the high-continuation state raises the bidding-round mean; conditioning on completed sales also changes the composition by removing no-sale histories.

and both standing-value kinks. Table OA5.2 reports the root box, Krawczyk image, and interval Jacobian. The residual system used for the local calculation is

$$\mathcal{E}(q) = (1 - e^{-10}) \begin{pmatrix} M_P(q)[T_P(q) - q_L] \\ M_K(q)[T_K(q) - q_H] \end{pmatrix}, \qquad J = D_q \mathcal{E}.$$

Since both action masses are positive, its zeros coincide with the boundary-price fixed points. Strict Krawczyk inclusion establishes a locally unique zero, and the determinant enclosure excludes zero. At a root, nonsingularity also holds for the Jacobian of $T(q) - q$, since the two residual systems differ by a nonsingular diagonal factor. This certifies the required price box at the displayed primitives, while the corollary's distributional neighborhood remains qualitative: no explicit perturbation radius is computed.

Table OA5.2: Root Enclosure and Local Uniqueness Certificate

Quantity	Outward enclosure
Root-box coordinate q_L	[0.609022919151066, 0.609022919153068]
Root-box coordinate q_H	[0.715349726667131, 0.715349726669133]
Krawczyk image, coordinate L	[0.609022919152066548242, 0.609022919152066548244]
Krawczyk image, coordinate H	[0.715349726668132175635, 0.715349726668132175636]
Jacobian entry J_{11}	[−0.154272567267, −0.154272567240]
Jacobian entry J_{12}	[0.053728588475, 0.053728588488]
Jacobian entry J_{21}	[−0.000041801644, −0.000041801637]
Jacobian entry J_{22}	[−0.135487742935, −0.135487742915]
$\det J$	[0.020904287872, 0.020904287879]

Notes: The calculation uses Arb through `python-flint 0.9.0`, at 60 decimal digits. These intervals are outward-rounded displays of the computed balls. Strict Krawczyk inclusion is checked before display rounding. The Jacobian is for the unnormalised zero-rent equations defined in the text, not for the price map T itself.

Table OA5.3: Uniform Price-Box and Equilibrium Margins

Quantity	Certified lower bound or range
$T_P(\ell_L, \ell_H) - \ell_L$	0.004000000
$u_L - T_P(u_L, u_H)$	0.003999999
$T_K(u_L, \ell_H) - \ell_H$	0.003996513
$u_H - T_K(\ell_L, u_H)$	0.003996068
$\min_{b\in[0,1],\, q\in\mathcal{Q}_0} D_L(b;q)$	0.020065585
$\min_{b\in[0,1],\, q\in\mathcal{Q}_0}[\bar{d} - D_H(b;q)]$	0.166323464
$\min_{b\in[0,1],\, q\in\mathcal{Q}_0} Z(b;q)$	0.011915854
$\ell_L - C_L(0)$	0.165580252
$C_L(1) - u_L$	0.051934414
$\ell_H - C_H(0)$	0.152820206
$C_H(1) - u_H$	0.038429512
$\min_b[C_H(b) - C_L(b)]$	0.068872090
$\mathbb{E}[C_H(b) \mid P] - q_L$	0.085924816
$q_H - \mathbb{E}[C_L(b) \mid K]$	0.086482870
$C_L(1) - q_L$	0.055934414
$C_H(1) - q_H$	0.042429513
Root-box range of D_L	[0.024065585, 0.709022920]
Root-box range of $\widetilde{D}_H$	[0.043981439, 0.765349727]
Root-box range of D_H	[0.063897294, 0.821676535]

Notes: The first eleven bounds hold on the entire price rectangle $\mathcal{Q}_0$. The state-ordering bound holds for all buyer values. Seller-response, D1 and cutoff bounds hold on the certified root box. The finite corner argument in Section OA5.2 extends the four face checks to whole faces; the cutoff identities reduce interiority to endpoint checks. All displayed lower bounds are rounded down.

OA5.4 Late-buyer preparation

For this extension of the ban comparison, write O for an offer attempt and A for acceptance, corresponding to I and A_c in the main paper.

Corollary OA5.1 (Late-buyer preparation)**.** *Let $S_s(\tau) \geq 0$ denote the object-specific due-diligence, financing, travel, and related preparation resources that remain unspent when an early offer is accepted at time τ. Holding participation and the terminal auction fixed,*

$$W^B - W^D = \mathbb{E}\Big[\kappa \mathbf{1}\{O\} + \mathbf{1}\{A\}\Big(L_s(b) - d_B - d_{S,s} - S_s(\tau)\Big)\Big]. \tag{OA5.6}$$

Preparation incurred before τ, or useful independently of this object, cancels from the comparison. Only object-specific resources that the early sale actually avoids make discretion more attractive.

Proof. Relative to Proposition 4.1, an accepted early sale saves exactly $S_s(\tau)$ in addition to the buyer's resolution benefit and the seller's benefit from early resolution. Costs already incurred are common to the two policy regimes and cancel. □

Corollary OA5.1 formalizes the timing margin emphasized in Finansdepartementet (2021, ch. 6.6.3) at fixed participation.

OA5.5 A probing calibration

For scale, take values $F = U[0,1]$, deterministic late participation $N = n_s$, and the aligned states $(n_L, v_L, d_{S,L}) = (2, 0.38, 0.02)$, $(n_H, v_H, d_{S,H}) = (3, 0.42, 0)$, with $\pi = 1/2$ and $\kappa = 0.02$. The exact waiting-only boundary is $\bar{D}_L = 0.02$. The highest-value buyer's probing–knockout cutoff is $\bar{D}_L + \Delta_D/\pi = 0.092821907$, giving the sufficient $\{W, P\}$ interval $0.02 < \bar{d} \leq 0.092821907$. At $\bar{d} = 0.045$, let the benefit from early resolution have the truncated exponential distribution $G(d) = (1 - e^{-10d})/(1 - e^{-10\bar{d}})$ on $[0, \bar{d}]$. The probing price is $q_L = 0.659155058$ and attempts occur with probability 0.073608826. No early offer, a rejected attempt, and successful pre-emption then have probabilities 0.926391174, 0.036804413, and 0.036804413.

OA5.6 Numerical Calibrations

Table OA5.4 summarizes the numerical calibrations. In each calibration based on the baseline model, the designated early buyer and late buyers share the listed value distribution. Write $\mathrm{TE}(\lambda; \bar{d})$ for the truncated-exponential distribution of resolution benefits in (OA5.3).

Table OA5.4: Numerical Calibrations

Exercise	Value distribution	Buyer heterogeneity	Other primitives and role
Waiting–probing regime	$F = U[0,1]$	$G = \text{TE}(10; .045)$	$(n_L, v_L, d_{S,L}) = (2, .38, .02)$, $(n_H, v_H, d_{S,H}) = (3, .42, 0)$, $\kappa = .02$; no offer, rejection, and completed pre-emption all occur.
Aligned full menu	$F = U[0,1]$	$G = \text{TE}(10; 1)$	The same three-channel aligned states, $\kappa = .05$; probing and knockout offers both occur.
Seller resolution benefits and D1 survival	$F = U[0,1]$	$G = \text{TE}(10; 1)$	Common $(n, v) = (2, .4)$, $(d_{S,L}, d_{S,H}) = (.02, 0)$, $\kappa = .01$; seller resolution benefits screen offers, and a $\{W, K\}$ equilibrium survives D1.
Ban comparison	F^S and F^T in the welfare comparison in the main paper	$G = 10^{-4}U[0,1] + (1 - 10^{-4})\,\text{Beta}(1, 30)$	$(n_L, v_L) = (2, .2)$, $(n_H, v_H) = (6, .4)$, $d_{S,L} = d_{S,H} = 0$, $\kappa = .02$; a ban lowers surplus under F^S and raises it under F^T.
Participation disclosure	$F = U[0,1]$	$G = \text{TE}(10; 1.2)$	$(n_L, n_H) = (1, 3)$, common $v = .4$, $d_{S,L} = d_{S,H} = 0$, $\kappa = .02$, with $\pi \in \{.65, .775\}$; disclosure raises surplus at $\pi = .65$ and lowers it at $\pi = .775$.
Post-bidding counteroffers	$F = U[0,1]$	$G = \text{TE}(10; .15)$	$(n_L, n_H) = (2, 3)$, common $v = .4$, $d_{S,L} = d_{S,H} = .01$, $\kappa = .01$; probing survives action-linked counteroffers.
Continued bargaining	$F = U[0,1]$	Atom–continuous G_η	Positive proposal costs; all three outcomes survive continued bargaining. The type distribution is described in Section OA4.1.
Buyer risk aversion	$F = U[0,1]$	$d_B = 0$, binary CARA coefficient	$(n_L, n_H) = (2, 7)$, common $v = 0$, $\kappa = .01$; risk aversion supports probing, which survives D1.

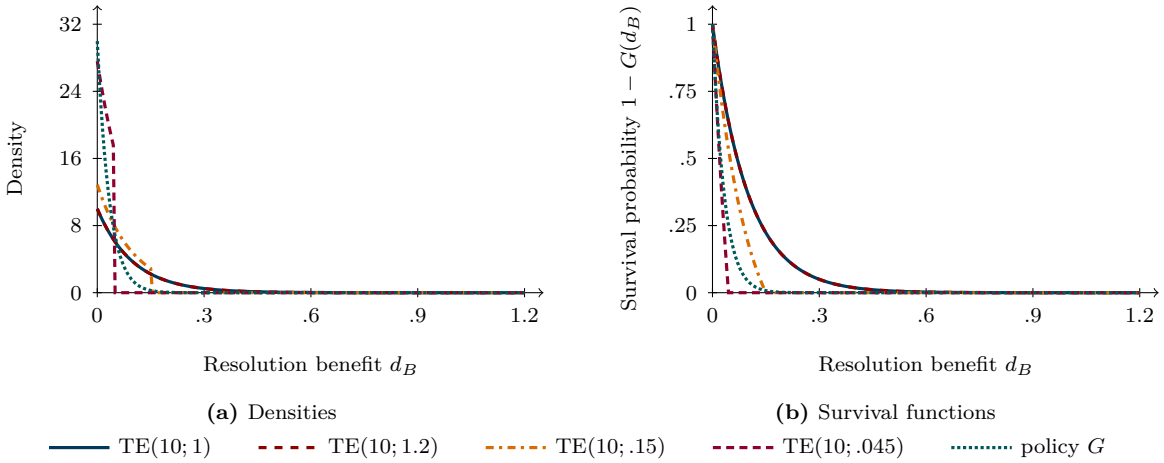


Figure OA5.1: Distributions of Benefits from Early Resolution

Notes: $\text{TE}(\lambda; \bar{d})$ denotes a rate-λ exponential truncated to $[0, \bar{d}]$. The policy distribution is $10^{-4}U[0,1] + (1 - 10^{-4})\,\text{Beta}(1, 30)$. The continued-bargaining specification contains an atom, and the risk-aversion exercise uses two discrete risk-aversion types; neither has a continuous density of resolution benefits to plot.

Calculation Workbook. The accompanying workbook gives code, inputs, outputs, and reproduction instructions in a single downloadable Jupyter notebook. It distinguishes interval certificates, exact-arithmetic checks, floating-point audits, and research diagnostics. The aligned-state certificate is reported in Section OA5.3.

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